

# HIGHER ADELES AND NON-ABELIAN RIEMANN-ROCH

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**ABSTRACT.** We show a Riemann-Roch theorem for group ring bundles over an arithmetic surface; this is expressed using the higher adeles of Beilinson-Parshin and the tame symbol via a theory of adelic equivariant Chow groups and Chern classes. The theorem is obtained by combining a group ring coefficient version of the local Riemann-Roch formula as in Kapranov-Vasserot with results on K-groups of group rings and an explicit description of group ring bundles over  $\mathbb{P}^1$ . Our set-up provides an extension of several aspects of the classical Fröhlich theory of the Galois module structure of rings of integers of number fields to arithmetic surfaces.

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## INTRODUCTION

In this paper we initiate an adelic theory of Galois module structure for arithmetic surfaces which extends many aspects of the corresponding theory for ring of integers of finite Galois extensions of number fields. Adelic methods have played an important role in the classical theory of Galois module structure ([17]); the starting point is Fröhlich's adelic description of the class group of finitely generated locally free modules for the integral group ring of a finite group. Here, we introduce into the picture the higher dimensional adeles of Beilinson and Parshin, certain “adelic Chow groups” defined using these, and also a host of other constructions, some of which are inspired from the theory of loop groups. Our main result is an adelic Riemann-Roch theorem for group ring bundles over an arithmetic surface; this can be used for the calculation of equivariant Euler characteristics of arithmetic surfaces with a finite group action.

To explain further we need to introduce some notation. Let  $Y$  be a projective regular arithmetic surface over  $\mathbb{Z}$ ; i.e the structure morphism  $Y \rightarrow \text{Spec}(\mathbb{Z})$  is projective and flat of relative dimension 1 and  $Y$  is regular and irreducible. Suppose that  $G$  is a finite group. By definition, an  $\mathcal{O}_Y[G]$ -bundle  $\mathcal{E}$  of rank  $n$  on  $Y$  is a coherent sheaf of (left)  $\mathcal{O}_Y[G]$ -modules which is locally free on  $Y$ , i.e there is a finite affine Zariski open cover  $Y = \cup_{i \in I} U_i$ ,  $U_i = \text{Spec}(A_i)$ , of  $Y$  such that  $\mathcal{E}|_{U_i}$  is the sheaf that corresponds to a free  $A_i[G]$ -module of rank  $n$ . To such an  $\mathcal{E}$  we can associate a projective Euler characteristic  $\chi^P(Y, \mathcal{E})$  in the Grothendieck group  $K_0(\mathbb{Z}[G])$  of finitely generated projective  $\mathbb{Z}[G]$ -modules as follows (see [6]). Consider the Čech complex  $C^\bullet(\{U_i\}, \mathcal{E})$  obtained from  $\mathcal{E}$  and the cover  $\{U_i\}$ ; one can show that  $C^\bullet(\{U_i\}, \mathcal{E})$  is a “perfect” complex of  $\mathbb{Z}[G]$ -modules, i.e. that there is a bounded complex  $(P^\bullet)$  of finitely generated projective  $\mathbb{Z}[G]$ -modules  $P^j$  and a  $\mathbb{Z}[G]$ -map of complexes  $P^\bullet \rightarrow C^\bullet(\{U_i\}, \mathcal{E})$  which induces an isomorphism on cohomology groups. Then we define

$$\chi^P(Y, \mathcal{E}) = \sum_j (-1)^j [P^j]$$

where  $[P^j]$  stands for the class of the module  $P^j$  in the Grothendieck group  $K_0(\mathbb{Z}[G])$ ; this is independent of the choice of the cover  $\{U_i\}$  and of the complex  $P^\bullet$ . Recall that by Swan [48] all finitely generated projective  $\mathbb{Z}[G]$ -modules are locally free. This gives a rank homomorphism  $\text{rank} : K_0(\mathbb{Z}[G]) \rightarrow \mathbb{Z}$  whose kernel  $K_0^{\text{red}}(\mathbb{Z}[G])$  can be identified with the class group  $\text{Cl}(\mathbb{Z}[G])$  of finitely generated locally free  $\mathbb{Z}[G]$ -modules studied by Fröhlich.

If  $G$  is abelian, we can consider  $\mathcal{E}$  as a vector bundle over the scheme  $Y \times G^*$  with  $G^* = \text{Spec}(\mathbb{Z}[G])$  the Cartier dual; the class group  $\text{Cl}(\mathbb{Z}[G])$  can be identified with the Picard group  $\text{Pic}(G^*)$ . In this case, versions of the Riemann-Roch theorem for  $Y \times G^* \rightarrow G^*$  (such as the Deligne-Riemann-Roch theorem of [13]) can be used to calculate the element  $\chi^P(Y, \mathcal{E}) - \chi^P(Y, \mathcal{O}_Y[G]^n)$  in  $\text{Cl}(\mathbb{Z}[G]) = \text{Pic}(G^*)$ . This basic observation together with the theory of cubic structures eventually leads to a satisfactory theory in this case, especially in the crucial case when the bundle  $\mathcal{E}$  is obtained from a tame cover  $X \rightarrow Y$  ([40], [10]; see also below). When  $G$  is not abelian, the above do not apply. Then, we will see that the adelic point of view gives a natural framework for developing a sufficiently fine theory that can be used to calculate the classes  $\chi^P(Y, \mathcal{E})$ .

Indeed, it is our point of view here that the bundle  $\mathcal{E}$  can also be described by adelic transition matrices as follows, where “adelic” is meant in the sense of the higher dimensional adeles of Beilinson and Parshin. Recall that a (non-degenerate) Parshin  $m$ -chain of  $Y$  is an ordered  $m$ -tuple  $\eta = (\eta_{i_1}, \dots, \eta_{i_m})$  of points of  $Y$  with  $i_1 < \dots < i_m$ , such that  $\eta_{i_k}$  lies on the Zariski closure of the previous point  $\eta_{i_{k-1}}$  and with the codimension of the closure of  $\eta_i$  in  $Y$  equal to  $i$ . Since  $Y$  is of dimension 2 we have  $m = 1, 2$  or  $3$ . For every such Parshin chain  $\eta$  one can define the “multicompletion”  $\hat{\mathcal{O}}_{Y,\eta} = \hat{\mathcal{O}}_{Y,\eta_{i_1} \dots \eta_{i_m}}$  by successively taking localizations and completions of  $\mathcal{O}_Y$  starting from  $\eta_{i_m}$  (see Proposition 1.2). For example, if  $\eta$  is a 1-chain and  $\eta$  is a single point,  $\hat{\mathcal{O}}_{Y,\eta}$  is the completion of the local ring of  $Y$  at  $\eta$ . In particular, for the generic point  $\eta_0$  of  $Y$  we have  $\hat{\mathcal{O}}_{Y,\eta_0} = K(Y)$ , the function field of  $Y$ . If  $\eta = (\eta_0, \eta_1, \eta_2)$  is a 3-chain then  $\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}$  is a finite direct sum of two-dimensional local fields. For each point  $\xi$  of  $Y$ , we can pick a  $\hat{\mathcal{O}}_{Y,\xi}[G]$ -basis  $e_\xi = \{e_\xi^h\}_{h=1}^n$  of the completed stalk  $\hat{\mathcal{E}}_\xi$ . If  $(\eta_0, \eta_1, \eta_2)$  is a Parshin triple, and  $0 \leq i < j \leq 2$ , then we can compare bases at  $\eta_i$  and  $\eta_j$  and write

$$e_{\eta_i} = \lambda_{\eta_i\eta_j} \cdot e_{\eta_j}, \quad \text{with } \lambda_{\eta_i\eta_j} \in \text{GL}_n(\hat{\mathcal{O}}_{Y,\eta_i\eta_j}[G]).$$

The matrices  $\lambda_{\eta_i\eta_j}$  are “adelic transition matrices” for the bundle  $\mathcal{E}$ . We say that  $\mathcal{E}$  has elementary structure if we can choose bases as above such that the corresponding transition matrices  $\lambda_{\eta_i\eta_j}$ , regarded in the infinite general linear group  $\text{GL}(\hat{\mathcal{O}}_{Y,\eta_i\eta_j}[G])$ , belong to the commutator subgroup  $E(\hat{\mathcal{O}}_{Y,\eta_i\eta_j}[G])$  generated by elementary matrices.

By an “adelic Riemann-Roch theorem” for  $\mathcal{E}$ , we mean a formula that allows us to calculate the Euler characteristic  $\chi^P(Y, \mathcal{E})$  starting from the adelic transition matrices  $\{\lambda_{\eta_i\eta_j}\}$  and which involves suitable “adelic characteristic classes” of  $\mathcal{E}$ .

Our main result gives an adelic Riemann-Roch theorem for bundles  $\mathcal{E}$  that have elementary structure, under some technical assumptions on  $Y$  and  $G$ . In particular, for this we will assume that the group algebra  $\mathbb{Q}[G]$  splits in the sense that we can write

$$(0.1) \quad \mathbb{Q}[G] = \prod_i \text{Mat}_{m_i \times m_i}(Z_i),$$

where each  $Z_i$  is a commutative finite field extension of  $\mathbb{Q}$ , i.e a number field. However, a number of the results in the paper are true for arbitrary finite groups  $G$ . Also, in addition to our standing hypotheses on  $Y$ , we assume:

(H) *All the irreducible components of the fibers of the morphism  $Y \rightarrow \text{Spec}(\mathbb{Z})$  are smooth (therefore also reduced) and furthermore, the fibers at primes that divide the order of the group  $G$  are irreducible.*

To describe the Riemann-Roch theorem we need to explain several important ingredients:

We first have the adelic Chow groups  $\text{CH}_{\mathbb{A}}^i(Y[G])$  for  $i = 1, 2$ . We define

$$\text{CH}_{\mathbb{A}}^2(Y[G]) := \frac{\prod'_{(\eta_0, \eta_1, \eta_2)} K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}[G]) \cdot \prod'_{(\zeta, \xi)} K_2(\hat{\mathcal{O}}_{Y,\zeta\xi}[G])^b}{\prod'_{(\zeta, \xi)} K_2(\hat{\mathcal{O}}_{Y,\zeta\xi}[G])^b}$$

as a quotient of a suitably restricted (adelic) products of  $K_2$ -groups of multicompletions, where the indices range over all Parshin 3-chains and all 2-chains respectively. (See §2.b,

2.c for details.) Similarly, we set

$$\mathrm{CH}_{\mathbb{A}}^1(Y[G]) := \frac{\prod'_{(\eta_0, \eta_1)} K_1(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1}[G])}{K_1(K(Y)[G]) \cdot \prod_{\eta_1} K_1(\hat{\mathcal{O}}_{Y, \eta_1}[G])^b}.$$

These definitions are interesting even when  $G$  is the trivial group. If  $G = \{1\}$ ,  $\mathrm{CH}_{\mathbb{A}}^1(Y[G]) \cong \mathrm{Pic}(Y)$ . In the case of the trivial group and when  $Y$  is a projective smooth surface over a field a similar construction of a second adelic Chow group has been considered by Osipov. We conjecture that his construction agrees with the one described in this paper. Osipov shows that, in this geometric non-equivariant case, his second adelic Chow group agrees with the classical Chow group  $\mathrm{CH}^2(Y)$  of codimension 2 cycles up to rational equivalence ([39]). On the other hand, recall that by Fröhlich's classical results we have a canonical isomorphism

$$\mathrm{Cl}(\mathbb{Z}[G]) \cong \frac{\prod'_p K_1(\mathbb{Q}_p[G])}{(K_1(\mathbb{Q}[G]) \prod_p K_1(\mathbb{Z}_p[G]))^b}.$$

This isomorphism allows us to identify the class group  $\mathrm{Cl}(\mathbb{Z}[G])$  with the first adelic Chow group  $\mathrm{CH}_{\mathbb{A}}^1(\mathrm{Spec}(\mathbb{Z})[G])$  of  $\mathrm{Spec}(\mathbb{Z})$ .

The second ingredient of our Riemann-Roch theorem is a pushdown (Gysin) homomorphism along  $f : Y \rightarrow \mathrm{Spec}(\mathbb{Z})$

$$f_* : \mathrm{CH}_{\mathbb{A}}^2(Y[G]) \rightarrow \mathrm{CH}_{\mathbb{A}}^1(\mathrm{Spec}(\mathbb{Z})[G]) = \mathrm{Cl}(\mathbb{Z}[G]).$$

This is constructed by assembling homomorphisms

$$f_{* \eta_0 \eta_1 \eta_2} : K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}[G]) \rightarrow K_1(\mathbb{Q}_p[G]),$$

where  $p$  is the characteristic of the closed point  $\eta_2$ , which are obtained using either the classical tame symbol or Kato's residue symbol. Showing that these homomorphisms produce a pushdown  $f_*$  between the adelic Chow groups is a subtle affair that involves using various reciprocity laws; the most difficult part is proving that the denominator in the definition of  $\mathrm{CH}_{\mathbb{A}}^2(Y[G])$  maps to the denominator in the Fröhlich description of  $\mathrm{Cl}(\mathbb{Z}[G])$ . This comes from considering the central extension (0.2) which we will describe below.

Finally, the third ingredients are the adelic Chern classes  $c_1(\mathcal{E})$  and  $c_2(\mathcal{E})$  of  $\mathcal{E}$ . The first Chern class  $c_1(\mathcal{E})$  is defined for an arbitrary  $\mathcal{O}_Y[G]$ -bundle  $\mathcal{E}$ : It is given as the class of  $\prod_{(\eta_0, \eta_1)} \mathrm{Det}(\lambda_{\eta_0 \eta_1})$  in  $\mathrm{CH}_{\mathbb{A}}^1(Y[G])$  where  $\lambda_{\eta_0 \eta_1}$  are adelic transition matrices as above and  $\mathrm{Det}(\lambda)$  stands for the class of a matrix  $\lambda \in \mathrm{GL}_n(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1}[G])$  in  $K_1(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1}[G])$ . The second Chern class  $c_2(\mathcal{E})$  is only defined when  $\mathcal{E}$  has an elementary structure. Recall the Steinberg extension

$$1 \rightarrow K_2(\hat{\mathcal{O}}_{Y, \eta}[G]) \rightarrow \mathrm{St}(\hat{\mathcal{O}}_{Y, \eta}[G]) \rightarrow \mathrm{E}(\hat{\mathcal{O}}_{Y, \eta}[G]) \rightarrow 1$$

with  $\mathrm{E}(\hat{\mathcal{O}}_{Y, \eta}[G])$  the elementary subgroup of the infinite general linear group  $\mathrm{GL}(\hat{\mathcal{O}}_{Y, \eta}[G])$ . To construct the second Chern class, we choose lifts  $\tilde{\lambda}_{\eta_i \eta_j}$  of the transition matrices  $\lambda_{\eta_i \eta_j}$  to the Steinberg group and consider

$$z(\tilde{\lambda})_{(\eta_0, \eta_1, \eta_2)} := \tilde{\lambda}_{\eta_0 \eta_2} \cdot (\tilde{\lambda}_{\eta_0 \eta_1})^{-1} \cdot (\tilde{\lambda}_{\eta_1 \eta_2})^{-1}$$

as an element in  $K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}[G])$ . By choosing the lifts  $\tilde{\lambda}_{\eta_i \eta_j}$  carefully, we can guarantee that the Steinberg cocycle  $z(\tilde{\lambda}) := (z(\tilde{\lambda})_{(\eta_0, \eta_1, \eta_2)})_{(\eta_0, \eta_1, \eta_2)}$  is “adelic”, i.e. lies in the numerator of the right hand side in the definition of  $\text{CH}_{\mathbb{A}}^2(Y[G])$  (See Proposition 6.4). Then the Chern class  $c_2(\mathcal{E})$  is given as the class of the element  $z(\tilde{\lambda})$  in  $\text{CH}_{\mathbb{A}}^2(Y[G])$ .

We are now ready to state our main result.

**Theorem 0.1.** *Assume that  $\mathbb{Q}[G]$  splits and that  $Y \rightarrow \text{Spec}(\mathbb{Z})$  is a regular arithmetic surface that satisfies (H). Then, if  $\mathcal{E}$  is an  $\mathcal{O}_Y[G]$ -bundle of rank  $n$  with elementary structure, we have*

$$\chi^P(Y, \mathcal{E}) - \chi^P(Y, \mathcal{O}_Y[G]^n) = -f_*(c_2(\mathcal{E})).$$

Let us remark here that if  $\mathcal{E}$  has an elementary structure, then  $c_1(\mathcal{E})$  is trivial; this then explains the shape of the identity above. Indeed, in this case of relative dimension 1, this agrees with the shape of the classical Grothendieck-Riemann-Roch formula for vector bundles of rank  $n$  with trivial determinant (see for example [13]). When  $Y$  is the projective line  $\mathbb{P}_{\mathbb{Z}}^1$  we can show a more general Riemann-Roch type result for arbitrary  $\mathcal{O}_Y[G]$ -bundles.

Interesting examples of bundles  $\mathcal{E}$  for which one can apply the Riemann-Roch formula are provided as follows. Suppose that  $q : X \rightarrow Y$  is a finite flat  $G$ -cover of the arithmetic surface  $Y$ ; one can see that if the ramification of  $q$  is *tame* and  $\mathcal{F}$  is a  $G$ -equivariant bundle on  $X$ , then  $\mathcal{E} = q_*(\mathcal{F})$  gives a  $\mathcal{O}_Y[G]$ -bundle on  $Y$ . Then  $\chi^P(Y, \mathcal{E})$  is equal to the equivariant projective Euler characteristic  $\chi^P(X, \mathcal{F})$  studied in [6], [10] and other articles. The “cubic method” of [10] provides a very effective way of calculating such Euler characteristics but with the crucial limitation that  $G$  is abelian. Here we are allowing more general finite groups and so Theorem 0.1 adds significantly to the tools currently available for the calculation of such Euler characteristics. We plan to elaborate on such applications in the future.

We will now give some more details about our techniques and discuss the proof of the Riemann-Roch theorem.

Important input is provided by certain central extensions which are arithmetic versions of a standard construction in the theory of loop groups and infinite dimensional Kac-Moody Lie algebras. Suppose that  $R$  is a commutative ring and consider the formal power series ring  $R[[t]]$  and the Laurent power series ring  $R((t)) = R[[t]][t^{-1}]$ . We define a central extension

$$(0.2) \quad 1 \rightarrow K_1(R[G]) \rightarrow \mathcal{H}(R((t))[G]) \rightarrow \text{GL}'(R((t))[G]) \rightarrow 1$$

where  $\text{GL}'(R((t))[G])$  is a subgroup of the infinite general linear group  $\text{GL}(R((t))[G])$  that contains the commutator  $E(R((t))[G])$ . This central extension is provided via the choice of a determinant theory on  $R[[t]][G]$ -lattices in  $R((t))[G]^n$  for  $n \geq 1$ . This notion has been introduced by Drinfeld and Kapranov. Set  $L_0 = R[[t]][G]^n$ . Recall that a  $R[[t]][G]$ -lattice  $L$  in  $R((t))[G]^n$  is a projective  $R[[t]][G]$ -submodule of  $R((t))[G]^n$  such that  $t^N L_0 \subset L \subset t^{-N} L_0$  for some  $N > 0$ . For us, a determinant theory is a suitable functor from a category of  $R[[t]][G]$ -lattices to the virtual category  $V(R[G])$  of projective finitely generated  $R[G]$ -modules. We can construct a determinant theory as follows: First construct an  $\mathcal{O}_{\mathbb{P}_R^1}[G]$ -bundle  $\mathcal{E}(L)$  over the projective line  $\mathbb{P}_R^1$  by gluing the (trivial) bundles corresponding to

the modules  $R[t^{-1}][G]^n$  over  $\mathbb{A}_R^1 = \text{Spec}(R[t^{-1}])$  and  $L$  over  $\mathbb{A}_R^1 = \text{Spec}(R[[t]])$  using the identification  $L \otimes_{R[[t]]} R((t)) \cong R[t^{-1}][G]^n \otimes_{R[t^{-1}]} R((t))$  provided by the inclusion  $L \subset R((t))[G]^n$ . Now we can consider the determinant of the cohomology complex in the derived category

$$\delta(L) := \det(\text{R}\Gamma(\mathbb{P}_R^1, \mathcal{E}(L)))$$

as an object in the virtual category  $V(R[G])$ . The association  $L \mapsto \delta(L)$  gives a determinant theory. For  $g \in \text{GL}'_n(R((t))[G])$ , we consider  $L = L_0 \cdot g^{-1}$ , so that  $\mathcal{E}(L)$  has  $g$  as a transition matrix along a formal neighborhood of  $t = 0$ . The central extension  $\mathcal{H}_n(R((t))[G])$  is a group with elements pairs  $(g, \phi_g)$  with  $g \in \text{GL}'_n(R((t))[G])$  and  $\phi_g$  an isomorphism between  $\delta(L_0)$  and  $\delta(L_0 \cdot g^{-1})$ . (From the very definition of  $\text{GL}'_n(R((t))[G])$  there exists such an isomorphism. See §3.d for details.) Now consider the direct limit as  $n$  goes to infinity to obtain (0.2).

By the universality of the Steinberg extension we obtain a map of central extensions

$$(0.3) \quad \begin{array}{ccccccc} 1 & \rightarrow & \text{K}_2(R((t))[G]) & \rightarrow & \text{St}(R((t))[G]) & \rightarrow & \text{E}(R((t))[G]) \rightarrow 1 \\ & & \partial \downarrow & & \partial \downarrow & & \downarrow \\ 1 & \rightarrow & \text{K}_1(R[G]) & \rightarrow & \mathcal{H}(R((t))[G]) & \rightarrow & \text{GL}'(R((t))[G]) \rightarrow 1. \end{array}$$

A first incarnation of the Riemann-Roch theorem in this case is the fact that  $\partial$  can be calculated using the tame symbol, in fact,  $\partial$  is equal to the inverse of the tame symbol when  $R$  is a field and  $G$  is trivial (see Proposition 3.8). In fact, this statement is often regarded as a “local” Riemann-Roch formula, see [29].

We can use this to obtain an adelic Riemann-Roch formula for bundles over  $\mathbb{P}^1 = \mathbb{P}_{\mathbb{Z}}^1$  as follows. First we show, by using an equivariant version of an argument of Horrocks, that each  $\mathcal{O}_{\mathbb{P}^1}[G]$ -bundle  $\mathcal{E}$  of rank  $n$  over  $\mathbb{P}^1$  which is trivial along the section  $(1 : 1)$  in homogeneous coordinates can be obtained by gluing trivial bundles over  $\text{Spec}(\mathbb{Z}[t])$  and  $\text{Spec}(\mathbb{Z}[t^{-1}])$  via a transition matrix  $g \in \text{GL}_n(\mathbb{Z}[t, t^{-1}][G])$ . If the bundle has, in addition, degree 0, then the matrix  $g$  regarded in  $\text{GL}_n(\mathbb{Q}[t, t^{-1}][G])$  and in  $\text{GL}_n(\mathbb{Z}_p[t, t^{-1}][G])$ , for each prime  $p$ , lies in the subgroups  $\text{GL}'_n(\mathbb{Q}[t, t^{-1}][G])$  and  $\text{GL}'_n(\mathbb{Z}_p[t, t^{-1}][G])$  respectively. By definition, this means that the base changes  $\delta(\mathcal{E})_{\mathbb{Q}}$  and  $\delta(\mathcal{E})_{\mathbb{Z}_p}$  of the determinant of cohomology  $\delta(\mathcal{E}) = \det(\text{R}\Gamma(\mathbb{P}^1, \mathcal{E}))$  are isomorphic, as elements in the virtual categories  $V(\mathbb{Q}[G])$  and  $V(\mathbb{Z}_p[G])$ , to the free rank  $n$  elements  $[\mathbb{Q}[G]^n]$  and  $[\mathbb{Z}_p[G]^n]$ ; suppose that  $\alpha_{\mathbb{Q}}$ ,  $\alpha_p$  are choices of corresponding isomorphisms. The pairs  $(g, \alpha_{\mathbb{Q}})$  and  $(g, \alpha_p)$  are then elements of  $\mathcal{H}_n(\mathbb{Q}((t))[G])$  and  $\mathcal{H}_n(\mathbb{Z}_p((t))[G])$ ; these elements lift  $g$  considered in  $\text{GL}'_n(\mathbb{Q}[t, t^{-1}][G])$  and  $\text{GL}'_n(\mathbb{Z}_p[t, t^{-1}][G])$  respectively. Both  $\alpha_{\mathbb{Q}}$  and  $\alpha_p$  induce isomorphisms between  $\delta(\mathcal{E})_{\mathbb{Q}_p}$  and  $[\mathbb{Q}_p[G]^n]$ ; by comparing them we obtain an element  $\alpha_p^{-1} \cdot \alpha_{\mathbb{Q}}$  of the automorphism group of  $[\mathbb{Q}_p[G]^n]$ , i.e an element of  $\text{K}_1(\mathbb{Q}_p[G])$ . The class  $\chi^P(\mathbb{P}^1, \mathcal{E}) - \chi^P(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}[G]^n)$  coincides with the class of  $\delta(\mathcal{E})$  in the class group  $\text{Cl}(\mathbb{Z}[G])$ ; by the above, this can now be obtained as the class of the element  $\prod_p \alpha_p^{-1} \cdot \alpha_{\mathbb{Q}} \in \prod_p' \text{K}_1(\mathbb{Q}_p[G])$ . The local Riemann-Roch formula that relates the central extensions above via the tame symbol will now eventually lead to a proof of our main theorem for  $\mathbb{P}^1$  but this still requires a fair amount of work. Indeed, first, we need to show that the bundles we are considering have, after suitable changes of basis, elementary transition matrices and therefore also a well-defined second Chern class



$c_2(\mathcal{E})$ . We also need to explain how to express a Steinberg cocycle that can be used to calculate  $c_2(\mathcal{E})$  in terms of the original transition matrix  $g$ ; notice that  $g$  itself might not be elementary.

The notion of elementary structure is, as it turns out, quite subtle. Observe that the transition matrix  $\lambda_{\eta_i \eta_j}$  is elementary when the class  $[\lambda_{\eta_i \eta_j}]$  in  $K_1(\hat{\mathcal{O}}_{Y, \eta_i \eta_j}[G])$  is trivial. Therefore, examining when adelic transition matrices are elementary involves the consideration of  $K_1$ -groups of group rings with coefficients in certain  $p$ -adically complete rings, as are some of the multicompletions considered above. For this we need to use the results of [9]. In particular, we can see that our notion of elementary structure is appropriately restrictive; for example, our considerations show that any  $\mathcal{O}_{\mathbb{P}^1}[G]$ -bundle which is trivial along  $(1 : 1)$  and has zero degree has an elementary structure. Considering these multicompletions also necessitates that we develop certain “ $p$ -adically completed” variations of the central extension (0.2); for example, we need such extensions for group rings with coefficients in the  $p$ -adic completion  $\mathbb{Z}_p\{\{t\}\} = \varprojlim_n \mathbb{Z}/p^n((t))$  of  $\mathbb{Z}_p((t))$  or in the two-dimensional local field  $\mathbb{Q}_p\{\{t\}\} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p\{\{t\}\}$ .

The above gives the rough idea of the proof of the Riemann-Roch theorem for  $Y = \mathbb{P}^1$ . To obtain the main result for an  $\mathcal{O}_Y[G]$ -bundle  $\mathcal{E}$  on a more general (regular) arithmetic surface  $Y \rightarrow \text{Spec}(\mathbb{Z})$  we argue as follows: By work of B. Green, there exists a finite flat morphism  $\pi : Y \rightarrow \mathbb{P}^1$ ; we use pushforward by  $\pi$  to reduce the proof to the case of  $\mathbb{P}^1$ . The fact that, as we assume,  $\mathcal{E}$  has elementary structure does not imply that this is also the case for  $\pi_*(\mathcal{E})$ ; this complicates the argument. However, we can still find a simple bundle  $\mathcal{V}$  with induced  $G$ -action so that the direct sum  $\pi_*(\mathcal{E}) \oplus \mathcal{V}$  is an  $\mathcal{O}_{\mathbb{P}^1}[G]$ -bundle with elementary structure on  $\mathbb{P}^1$ . We now explicitly relate Steinberg cocycles that compute the second Chern class of  $\mathcal{E}$  with corresponding Steinberg cocycles that compute the second Chern class of  $\pi_*(\mathcal{E}) \oplus \mathcal{V}$  on  $\mathbb{P}^1$  and there is a resulting identity (Proposition 8.6) that relates the second Chern classes of  $\mathcal{E}$  and of  $\pi_*(\mathcal{E}) \oplus \mathcal{V}$ . This identity can be viewed as an adelic Riemann-Roch formula for the finite flat morphism  $\pi$ . These considerations allow us to reduce the general case to the case  $Y = \mathbb{P}^1$  which is handled as explained above.

Our definitions of the second adelic Chern class and of the Gysin map were initially inspired by the work of Parshin in [42], [43] and of Osipov in [39]. The reader can also find similar or related constructions in the work of Hübl-Yekutieli [28] and Morrow [34]. Let us remark here that although our main interest in this paper is to the case of bundles for a group ring, our techniques can also provide interesting new results when the group  $G$  is trivial and even in the context of these references. Indeed, the current paper also contributes to the theme of refined Riemann-Roch type theorems; examples of such theorems are Deligne’s functorial Riemann-Roch theorem for relative curves [13], or the second author’s integral Grothendieck-Riemann-Roch theorem [41]. For example, we can consider vector bundles  $\mathcal{E}$  over an arithmetic surface  $Y \rightarrow \text{Spec}(R)$  where  $R$  is a Dedekind ring with finite residue fields, as is the ring of integers of a number field. Our methods can then be used to show an adelic Riemann-Roch theorem for  $f : Y \rightarrow \text{Spec}(R)$  and  $\mathcal{E}$  by factoring  $f$  as a composition of a finite flat morphism  $\pi : Y \rightarrow \mathbb{P}_R^1$  with the projection  $h : \mathbb{P}_R^1 \rightarrow \text{Spec}(R)$  and proving as above Riemann-Roch identities for  $\pi$  and  $h$ . This is an alternative to Grothendieck’s strategy of

proving Riemann-Roch by factoring  $f$  as a composition of a closed immersion followed by a projective bundle. The details will appear elsewhere. Indeed, it seems that Grothendieck's technique cannot be easily adapted to show an adelic Riemann-Roch theorem. However, it is plausible that, for higher dimensional varieties, an approach as above which starts by using [8] that provides a finite flat morphism to projective space could work instead.

We will now briefly describe the structure of the paper. In §1, we explain the theory of higher dimensional adeles of Beilinson and Parshin and give examples of the corresponding multicompletions for the case of arithmetic surfaces. In §2, we give the definitions of the adelic Chow groups. The constructions of the central extensions (0.2) and of its  $p$ -adically complete variants are given in §3. In the same paragraph, we also show that the corresponding maps  $\partial$  (resp.  $\hat{\partial}$  in the  $p$ -adic variant) in (0.3) are given via the inverse of the tame symbol (resp. of Kato's residue symbol). In §4, we define the pushdown maps  $f_{*\eta_0\eta_1\eta_2}$  and show that they induce a Gysin map  $f_*$  between the adelic Chow groups as above. In §5, we explain the formalism of adelic transition matrices and give the definition of the first adelic Chern class. The notion of elementary structure and the definition of the second adelic Chern class for bundles with elementary structure is given in §6. In §7 we state the main theorem and in §8 we explain the reduction of the proof to the case of bundles over  $\mathbb{P}^1$  by working with pushdown along a finite flat morphism  $Y \rightarrow \mathbb{P}^1$ . Finally, the proof of the adelic Riemann-Roch identity for bundles over  $\mathbb{P}^1$  occupies §9.

## 1. BEILINSON-PARSHIN ADELES ON A SURFACE

**1.a. Parshin tuples and multicompletions.** In this section we will let  $Y$  be an irreducible separated Noetherian scheme of dimension  $d$ . We will recall the theory of adeles for  $Y$  developed by Parshin and Beilinson; see [42], [43], [2], [27], [50] and the useful survey [35] for more detailed accounts.

Following [2], let  $P(Y)$  be the set of points of  $Y$ . If  $\eta, \eta' \in P(Y)$  we will say that  $\eta \geq \eta'$  if  $\eta'$  is a point on the closure  $\bar{\eta}$  of  $\eta$ . Let  $S(Y)$  be the simplicial set associated to  $P(Y)$  and this order relation. Thus the  $n$ -simplex  $S(Y)_n$  is the set of all Parshin  $n+1$ -tuples  $(\eta(0), \dots, \eta(n))$  of points on  $Y$ , these being ordered sequences of  $n+1$  points in  $P(Y)$  such that  $\eta(0) \geq \eta(1) \geq \dots \geq \eta(n)$ . We will call such an  $n+1$ -tuple degenerate if  $\eta(i) = \eta(i+1)$  for some  $i$ ; otherwise it is non-degenerate. We will use the convention that a subscript on a point indicates its codimension on  $Y$ . Thus  $\eta_0$  is the generic point. The Parshin 1-tuples thus have the form  $(\eta_i)$  for some  $0 \leq i \leq d$ , the Parshin 2-tuples have the form  $(\eta_i, \eta_j)$  for some  $0 \leq i \leq j \leq d$ , and so on.

Suppose  $K_n$  is a subset of  $S(Y)_n$  and that  $\eta$  is a point of  $Y$ . Let  $\mathcal{O}_\eta = \mathcal{O}_{Y,\eta}$  be the local ring of  $Y$  at  $\eta$ , with maximal ideal  $m_\eta = m_{Y,\eta}$ . Let  $j_\eta : \text{Spec}(\mathcal{O}_\eta) \rightarrow Y$  be the natural morphism of schemes. If  $M$  is a module for  $\mathcal{O}_\eta$ , we also use  $M$  to denote both the quasi-coherent sheaf  $\mathcal{M}$  on  $\text{Spec}(\mathcal{O}_\eta)$  associated to  $M$  and the quasi-coherent sheaf  $(j_\eta)_*(\mathcal{M})$  on  $Y$ . In particular, the support of  $M$  as a sheaf on  $Y$  is contained in the closure  $\bar{\eta}$  of  $\eta$ . Define

$$(1.1) \quad {}_\eta K_{n-1} = \{(\eta(1), \eta(2), \dots, \eta(n)) \in S(Y)_{n-1} : (\eta, \eta(1), \dots, \eta(n)) \in K_n\}.$$



**Definition 1.1.** *As  $n$  and  $K_n$  vary, there is a unique family of functors  $A(K_n, \bullet)$  from the category of quasi-coherent  $\mathcal{O}_Y$ -modules to the category of abelian groups for which the following is true:*

1.  $A(K_n, \bullet)$  commutes with direct limits.
2. Suppose  $M$  is a coherent  $\mathcal{O}_Y$ -module.
  - a. If  $n = 0$ , then

$$(1.2) \quad A(K_n, M) = A(K_0, M) = \prod_{\eta \in K_0} \varprojlim_{\ell} (M \otimes_{\mathcal{O}_Y} (\mathcal{O}_{\eta}/m_{\eta}^{\ell})).$$

- b. If  $n > 0$ , then

$$(1.3) \quad A(K_n, M) = \prod_{\eta \in P(Y)} \varprojlim_{\ell} A({}_{\eta}K_{n-1}, M \otimes_{\mathcal{O}_Y} (\mathcal{O}_{\eta}/m_{\eta}^{\ell})).$$

A subtlety in this definition is that the sheaf  $M \otimes_{\mathcal{O}_Y} (\mathcal{O}_{\eta}/m_{\eta}^{\ell})$  appearing on the right side of (1.3) will not in general be coherent. Thus one must calculate the value of  $A({}_{\eta}K_n, \bullet)$  on the latter sheaf by taking an inductive limit.

When  $K_n = \{(\eta(0), \dots, \eta(n))\}$  consists of a single non-degenerate Parshin chain of length  $n + 1$  and  $M = \mathcal{O}_Y$ , we will denote by

$$(1.4) \quad \hat{\mathcal{O}}_{Y, \eta(0)\eta(1)\dots\eta(n)} = A(K_n, \mathcal{O}_Y)$$

the corresponding multicompletion of  $\mathcal{O}_Y$ .

**1.b. Examples of multicompletions.** Suppose here that  $K_n = \{(\eta(0), \dots, \eta(n))\}$  consists of a single non-degenerate Parshin chain of length  $n + 1$ . Let  $\text{Spec}(R)$  be an open affine subset of  $Y$  which contains  $\eta(0)$ . Then, for all  $i$ ,  $\eta(i)$  corresponds to a prime ideal of  $R$ . Suppose  $\mathfrak{a}$  and  $\mathfrak{p}$  are ideals of  $R$ ,  $\mathfrak{p}$  is prime and that  $N$  is an  $R$ -module. As in [27, p. 250], let  $S_{\mathfrak{p}}^{-1}N$  be the localization of  $N$  at  $S_{\mathfrak{p}} = R - \mathfrak{p}$  and define  $C_{\mathfrak{a}}N = \varprojlim_n N/\mathfrak{a}^n N$ .

The following result is shown by Huber in [27, Prop. 3.2.1].

**Proposition 1.2.** *Let  $M$  be a quasi-coherent  $\mathcal{O}_Y$ -module, and suppose that the restriction of  $M$  to  $\text{Spec}(R)$  is the sheaf associated to the  $R$ -module  $N$ . Then  $C_{\eta(0)}S_{\eta(0)}^{-1} \dots C_{\eta(n)}S_{\eta(n)}^{-1}R = B$  is a flat Noetherian  $R$ -algebra, and there is a natural isomorphism*

$$(1.5) \quad A(K_n, M) \cong B \otimes_R N.$$

*If  $M$  is coherent, so that  $N$  is Noetherian, one has*

$$(1.6) \quad A(K_n, M) \cong C_{\eta(0)}S_{\eta(0)}^{-1} \dots C_{\eta(n)}S_{\eta(n)}^{-1}N.$$

We now specialize further to the case in which  $M = \mathcal{O}_Y$  as in (1.4).

**1.b.1. Some Parshin chains of length 1.** We suppose in this subsection that  $n = 0$  and  $K_n = K_0 = \{(\eta(0))\}$  for a point  $\eta(0) = \eta_i$  of codimension  $i$  on  $Y$ . Then (1.6) shows that

$$\hat{\mathcal{O}}_{Y, \eta(0)} = \hat{\mathcal{O}}_{Y, \eta_i} = C_{\eta_i}S_{\eta_i}^{-1}R$$

is the completion of the local ring  $\mathcal{O}_{Y, \eta_i}$  at the powers of its maximal ideal.

We now suppose further that  $Y$  is irreducible, normal and flat over  $\mathbb{Z}$ , and that  $\eta_i = \eta_1$  has codimension 1. Then  $\hat{\mathcal{O}}_{Y,\eta_1}$  is a complete discrete valuation ring (dvr) of characteristic 0 with residue field  $k(\eta_1)$  given by the function field of the irreducible divisor  $\overline{\eta_1}$ . Let  $t$  be a uniformizer in  $\hat{\mathcal{O}}_{Y,\eta_1}$ .

If  $\eta_1$  is horizontal then  $k(\eta_1)$  has characteristic 0 and transcendence degree  $\dim(Y) - 2$  over  $\mathbb{Q}$ . In this case Hensel's Lemma shows there is an algebra homomorphism  $k(\eta_1) \rightarrow \hat{\mathcal{O}}_{Y,\eta_1}$  which is a section of the residue map  $\hat{\mathcal{O}}_{Y,\eta_1} \rightarrow k(\eta_1)$  and that  $\hat{\mathcal{O}}_{Y,\eta_1}$  is isomorphic to the formal power series ring  $k(\eta_1)[[t]]$ .

Suppose now that  $\eta_1$  is vertical, and let  $p$  be the prime of  $\mathbb{Z}$  determined by  $\eta_1$ . Recall that if  $A \rightarrow B$  is a local homomorphism between two local Noetherian rings such that  $B$  is complete and flat over  $A$  and  $B/m_B$  is a separable extension of  $A/m_A$ , then  $B$  is called a Cohen algebra over  $A$ . By [22, Chap. 0<sub>IV</sub>, 19.7.2],  $B$  is determined by its residue field  $B/m_B$  if  $m_B = Bm_A$ . We have assumed  $Y$  is flat over  $\mathbb{Z}$ . Hence if  $pB = m_B$  then  $B = \hat{\mathcal{O}}_{Y,\eta_1}$  is the Cohen algebra over  $A = \mathbb{Z}_p$  associated to  $k(\eta_1)$ . The statement that  $pB = m_B$  is equivalent to the statement that the closure of  $\eta_1$  has multiplicity 1 in the fiber of  $Y$  over  $p$ . In this case, one can describe  $\hat{\mathcal{O}}_{Y,\eta_1}$  explicitly by choosing a set theoretic section  $s$  for the residue map  $\hat{\mathcal{O}}_{Y,\eta_1} \rightarrow k(\eta_1)$ . Using  $s$  and  $t$  one can identify elements of  $\hat{\mathcal{O}}_{Y,\eta_1}$  with formal power series in  $t$  with coefficients in  $k(\eta_1)$ . The addition and multiplication laws of such series are then determined by the choice of  $t$  and  $s$ .

**1.b.2. Some Parshin chains of length 2.** We suppose in this subsection that  $n = 1$  and that  $Y$  is regular, quasi-projective and flat over  $\mathbb{Z}$ . As a result, all the local rings of  $Y$  are excellent. Let  $K_n = \{(\eta(0), \eta(1))\}$  consist of a Parshin chain of length 2. If  $\eta(0)$  is the generic point  $\eta_0$  of  $Y$ , then  $\eta(1)$  may be a point  $\eta_i$  of arbitrary codimension  $i \geq 1$ . The functor  $C_{\eta_0}$  is the identity functor, so (1.6) shows

$$\hat{\mathcal{O}}_{Y,\eta(0)\eta(1)} = \hat{\mathcal{O}}_{Y,\eta_0\eta_i} = (K(Y) - \{0\})^{-1} \hat{\mathcal{O}}_{Y,\eta_i}$$

where  $K(Y)$  is the function field of  $Y$ .

The other case in which  $n = 1$  which will be relevant to us is when, in addition to the above assumptions,  $Y$  is of dimension 2,  $\eta(0)$  is a codimension 1 point  $\eta_1$  on  $Y$  and  $\eta(1)$  is a closed point  $\eta_2$  on the closure of  $\eta_1$ . The local ring  $\mathcal{O}_{Y,\eta_2}$  and its completion  $\hat{\mathcal{O}}_{Y,\eta_2}$  are then two-dimensional UFD's. A local equation  $\pi_1 \in \mathcal{O}_{Y,\eta_2}$  for  $\eta_1$  factors in  $\hat{\mathcal{O}}_{Y,\eta_2}$  into the product  $\pi_1 = u \prod_{\alpha=1}^m t_\alpha^{b_\alpha}$  of a unit  $u \in \hat{\mathcal{O}}_{Y,\eta_2}^\times$  together with positive integral powers of non-associate irreducibles  $t_\alpha \in \hat{\mathcal{O}}_{Y,\eta_2}$ . These  $t_\alpha$  define the analytic branches at  $\eta_2$  of the closure of  $\eta_1$ . Notice that since  $\mathcal{O}_{Y,\eta_2}/(\pi_1)$  is a reduced excellent local ring, the same is true for its completion which can be identified with  $\hat{\mathcal{O}}_{Y,\eta_2}/(\pi_1) = \hat{\mathcal{O}}_{Y,\eta_2}/(\prod_{\alpha=1}^m t_\alpha^{b_\alpha})$ . This implies that  $b_\alpha = 1$ , for all  $\alpha$ , and we have

$$(1.7) \quad \pi_1 = u \prod_{\alpha=1}^m t_\alpha.$$

Let  $B_\alpha$  be the discrete valuation ring which is the completion of the localization of  $\hat{\mathcal{O}}_{Y,\eta_2}$  at the codimension one prime ideal generated by  $t_\alpha$ . Let  $p > 0$  be the residue characteristic of  $\eta_2$ . The residue ring  $R_\alpha = \hat{\mathcal{O}}_{Y,\eta_2}/(t_\alpha)$  is a complete local integral domain of dimension

1 with finite residue field  $k(\eta_2)$  of characteristic  $p$ . The fraction field of  $R_\alpha$  is the residue field  $k(B_\alpha)$  of  $B_\alpha$ . We will also use the notation  $k(\eta_{1,\alpha})$  for  $k(B_\alpha)$  in order to emphasize its dependence on  $\eta_1$ . The integral closure  $R'_\alpha$  of  $R_\alpha$  in  $k(\eta_{1,\alpha})$  is finite over  $R_\alpha$ . Hence  $tR'_\alpha \subset R_\alpha$  for some  $0 \neq t \in R_\alpha$ , so since  $R_\alpha/R_\alpha t$  is a finite ring, a power of the radical of  $R'_\alpha$  lies in  $R_\alpha$ . Thus  $R'_\alpha$  is complete with respect to the powers of its radical because  $R_\alpha$  is complete. It follows that  $R'_\alpha$  is local because it is an integral domain. We conclude that  $R'_\alpha$  is a complete discrete valuation ring with finite residue field. Thus  $k(B_\alpha) = k(\eta_{1,\alpha})$  is a local field of dimension 1 with finite residue field. We distinguish two cases:

- **$\eta_1$  is horizontal:** Then  $k(\eta_{1,\alpha})$  is isomorphic to a finite extension of  $\mathbb{Q}_p$ . By Hensel's Lemma,  $B_\alpha$  is isomorphic to the formal power series ring  $\mathbb{Q}_p(\eta_{1,\alpha})[[t_\alpha]]$ .
- **$\eta_1$  is vertical:** Then  $k(\eta_{1,\alpha})$  is the completion of a global function field at a closed point. Let  $t$  be an element of  $B_\alpha$  which has image equal to a uniformizer  $\bar{t}$  in the discretely valued field  $k(\eta_{1,\alpha})$ . Then  $k(\eta_{1,\alpha})$  is isomorphic to the Laurent formal power series field  $k(\eta_2)((\bar{t}))$ .

**Lemma 1.3.** *Suppose  $\eta_1$  is vertical. The maximal ideal  $B_\alpha t_\alpha$  of  $B_\alpha$  equals  $B_\alpha p$  if and only if  $\eta_1$  occurs with multiplicity 1 in the fiber of  $Y$  over  $p$ . In this case,  $B_\alpha$  is isomorphic to the Cohen ring over  $\mathbb{Z}_p$  having residue field  $k(\eta_2)((\bar{t}))$ . This is true, in particular, if the fiber of  $Y$  over  $p$  is smooth.*

*Proof.* In  $\mathcal{O}_{Y,\eta_2}$  one has a factorization

$$(1.8) \quad p = v \cdot \prod_{i=1}^j \pi_i^{a_i}$$

in which  $v \in \mathcal{O}_{Y,\eta_2}^\times$  is a unit,  $j \geq 1$ ,  $\pi_1$  is our chosen local equation for  $\overline{\eta_1}$  and the  $\pi_i$  are non-associate irreducibles. The multiplicity of  $\eta_1$  in the fiber of  $Y$  is 1 if and only if  $p$  is a uniformizer in the local ring  $\mathcal{O}_{Y,\eta_1} = (\mathcal{O}_{Y,\eta_2} - \mathcal{O}_{Y,\eta_2}\pi_1)^{-1}\mathcal{O}_{Y,\eta_2}$ . This is the case if and only if  $a_1 = 1$ . If  $2 \leq i \leq j$  then  $\mathcal{O}_{Y,\eta_2}/(\pi_1, \pi_i)$  is a finite discrete quotient of  $\mathcal{O}_{Y,\eta_2}$ , so  $(\pi_1, \pi_i)$  contains a positive power of the maximal ideal of  $\mathcal{O}_{Y,\eta_2}$ . Hence  $\hat{\mathcal{O}}_{Y,\eta_2}/\hat{\mathcal{O}}_{Y,\eta_2}(\pi_1, \pi_i)$  is finite, so (1.7) implies  $\pi_i$  has valuation 0 in  $B_\alpha$  when  $2 \leq i \leq j$  because  $\hat{\mathcal{O}}_{Y,\eta_2}/\hat{\mathcal{O}}_{Y,\eta_2}t_\alpha$  is infinite. Thus (1.7) and (1.8) show that  $p$  has valuation  $a_1$  with respect to the discrete valuation of  $\hat{\mathcal{O}}_{Y,\eta_2}$  associated to  $\alpha$ . Thus  $B_\alpha t_\alpha = B_\alpha p$  if and only if  $a_1 = 1$ , and this proves the first assertion in Lemma 1.3. The second is a consequence of the results about Cohen rings cited in §1.b.1. The last assertion is clear from the first.  $\square$

We can make the isomorphism in Lemma 1.3 more explicit in the following way. Define  $W(k(\eta_2))$  to be the ring of infinite Witt vectors over the finite residue field  $k(\eta_2)$ . Let  $W(k(\eta_2))\{\{t\}\}$  be the ring of all doubly infinite formal power series

$$\sum_{n=-\infty}^{\infty} a_n t^n$$

in which  $a_n \in W(k(\eta_2))$  and  $\lim_{n \rightarrow -\infty} a_n = 0$  in the  $p$ -adic topology on  $W(k(\eta_2))$ . Viewing  $k(\eta_2)$  as a finite subfield of the residue field  $k(\eta_2)((\bar{t}))$  of  $B_\alpha$ , we can take Teichmüller lifts of elements of  $k(\eta_2)$  to  $B_\alpha$  via the usual limit process. This produces a canonical algebra

embedding of  $W(k(\eta_2))$  into  $B_\alpha$ . There is then a unique topological ring isomorphism from  $W(k(\eta_2))\{\{t\}\}$  to  $B_\alpha$  which extends this embedding and sends  $t$  to itself as an element of  $B_\alpha$ .

We now return to the more general case in which we assume only that  $n = 2$ ,  $Y$  is regular, quasi-projective and flat over  $\mathbb{Z}$  of dimension 2,  $\eta(0)$  is a codimension 1 point  $\eta_1$  on  $Y$  and  $\eta(1)$  is a closed point  $\eta_2$  on the closure of  $\eta_1$ .

**Lemma 1.4.** *With the above notations, the ring homomorphism  $\mu$*

$$(1.9) \quad \hat{\mathcal{O}}_{Y, \eta(0)\eta(1)} = \hat{\mathcal{O}}_{Y, \eta_1\eta_2} = C_{\eta_1} S_{\eta_1}^{-1} \hat{\mathcal{O}}_{Y, \eta_2} \xrightarrow{\mu} \prod_{\alpha=1}^m B_\alpha$$

*resulting from (1.6) is an isomorphism. If  $\eta_1$  is horizontal we have*

$$(1.10) \quad \prod_{\alpha=1}^m B_\alpha \cong \bigoplus_{\alpha=1}^m \mathbb{Q}_p(\eta_{1\alpha})[[t_\alpha]].$$

*Suppose  $\eta_1$  is vertical and has multiplicity one in the fiber of  $Y$  over  $p$ . Then*

$$(1.11) \quad \prod_{\alpha=1}^m B_\alpha \cong \bigoplus_{\alpha=1}^m W(k(\eta_2))\{\{t_\alpha\}\}.$$

*Proof.* Statements (1.10) and (1.11) follow from (1.9) and the above computations of the  $B_\alpha$ . To show (1.9) it will suffice to prove the following. Fix  $\alpha$ , and let  $\tau_\alpha$  and  $\beta_\alpha$  be elements of  $\hat{\mathcal{O}}_{Y, \eta_2}$  such that  $\beta_\alpha \notin B_\alpha t_\alpha$ . Then  $\tau_\alpha/\beta_\alpha$  defines an element  $\overline{\tau_\alpha/\beta_\alpha}$  of the residue field  $k(B_\alpha)$  of  $B_\alpha$ . It will suffice to show that there is an element of the image of  $\mu$  whose component at  $B_\alpha$  has image  $\overline{\tau_\alpha/\beta_\alpha}$  in  $k(B_\alpha)$  and whose component at  $B_k$  for  $k \neq \alpha$  is a non-unit. The element

$$z = \prod_{k \neq \alpha, k=1}^m t_k$$

of  $\hat{\mathcal{O}}_{Y, \eta_2}$  has non-zero image  $\bar{z}$  in the one-dimensional local ring  $R_\alpha = \hat{\mathcal{O}}_{Y, \eta_2}/t_\alpha \hat{\mathcal{O}}_{Y, \eta_2}$ . Since the image of the ring  $\mathcal{O}_{Y, \eta_2}$  in the completion  $\hat{\mathcal{O}}_{Y, \eta_2}$  is dense, there is an element  $w \in \mathcal{O}_{Y, \eta_2}$  such that  $w$  and  $\beta_\alpha z$  generate the same ideal in  $R_\alpha$ . Thus there is an element  $u \in \hat{\mathcal{O}}_{Y, \eta_2}$  whose image in  $R_\alpha$  is a unit such that  $wu$  and  $\beta_\alpha z$  have the same image in  $R_\alpha$ . This  $u$  must be a unit of  $\hat{\mathcal{O}}_{Y, \eta_2}$ . Now  $w$  has non-zero image in  $R_\alpha$ , so  $w$  must be an element of  $\mathcal{O}_{Y, \eta_1}$  which is not in the maximal ideal of  $\mathcal{O}_{Y, \eta_1}$ . Thus  $w^{-1}u^{-1}\tau_\alpha z$  lies in  $S_{\eta_1}^{-1}\hat{\mathcal{O}}_{Y, \eta_2}$  and has image  $\overline{\tau_\alpha/\beta_\alpha}$  in  $k(B_\alpha) = \text{Frac}(R_\alpha)$ . Now  $w^{-1}u^{-1}\tau_\alpha z \in \hat{\mathcal{O}}_{Y, \eta_1\eta_2}$  because of the second equality in (1.9), so we have constructed the desired element. It follows that (1.9) is an isomorphism.  $\square$

**1.b.3. Some Parshin chains of length 3.** The last special case we will discuss is when  $Y$  is regular and integral of dimension  $n = 2$ . Let  $K_2 = \{(\eta(0), \eta(1), \eta(2))\} = \{(\eta_0, \eta_1, \eta_2)\}$  with  $\eta(0) = \eta_0$  the generic point of  $Y$ ,  $\eta(1) = \eta_1$  a codimension 1 point and  $\eta(2) = \eta_2$  a closed point on the closure of  $\eta_1$ . We find from (1.6) that

$$(1.12) \quad \hat{\mathcal{O}}_{Y, \eta(0)\eta(1)\eta(2)} = \hat{\mathcal{O}}_{Y, \eta_0\eta_1\eta_2} = (K(Y) - \{0\})^{-1} \hat{\mathcal{O}}_{Y, \eta_1\eta_2}$$

where  $\hat{\mathcal{O}}_{Y,\eta_1\eta_2}$  is a product of discrete valuation rings  $B_\alpha$  of the kind described above for the pair  $(\eta_1, \eta_2)$ . Since a uniformizer in  $B_\alpha$  divides the image in  $B_\alpha$  of an element of  $K(Y)$ , we find that that  $\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}$  is the product of the fraction fields of the  $B_\alpha$ .

1.b.4. *Base extensions.* In this section we suppose that  $h : X \rightarrow Y$  is a finite flat morphism of regular projective connected flat schemes over  $\mathbb{Z}$  of dimension 2. Let  $[X : Y]$  be the degree of  $h$ . Then  $h$  induces a map of simplicial sets  $h : S(X) \rightarrow S(Y)$ . The following result will be used in §8.

**Proposition 1.5.** *For all Parshin chains  $\eta$  in  $S(Y)$ , the homomorphism  $\mathcal{O}_Y \rightarrow h_*\mathcal{O}_X$  of sheaves of rings gives an isomorphism*

$$(1.13) \quad \mathcal{O}_X \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{Y,\eta} \rightarrow \bigoplus_{\eta' \in h^{-1}(\eta)} \hat{\mathcal{O}}_{X,\eta'}$$

*of free  $\hat{\mathcal{O}}_{Y,\eta}$ -modules of rank  $[X : Y]$ .*

*Proof.* Suppose first that  $\eta = \eta(0)$  consists of a single point of  $Y$ . Then  $\hat{\mathcal{O}}_{Y,\eta}$  is just the completion  $\hat{\mathcal{O}}_{Y,\eta(0)}$  of  $Y$  at  $\eta(0)$ , so (1.13) is clear from the fact that  $h : X \rightarrow Y$  is finite and flat of degree  $[X : Y]$ .

Suppose next that (1.13) holds for some  $\eta = (\eta(0), \dots, \eta(n))$  and that  $\eta(0)$  is not the generic point  $\eta_Y$  of  $Y$ . We now show (1.13) holds when  $\eta$  is replaced by  $\eta^* = (\eta_Y, \eta(0), \dots, \eta(n))$ , where  $h^{-1}(\eta^*) = \{(\eta_X, \eta') : \eta' \in h^{-1}(\eta)\}$ . By Proposition 1.2,

$$\hat{\mathcal{O}}_{Y,\eta^*} = K(Y) \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{Y,\eta}$$

when  $K(Y) = \mathcal{O}_{Y,\eta_Y}$  is the function field of  $Y$ . Thus since we assumed (1.13) is an isomorphism,

$$(1.14) \quad \begin{aligned} \mathcal{O}_X \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{Y,\eta^*} &= K(Y) \otimes_{\mathcal{O}_Y} (\mathcal{O}_X \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{Y,\eta}) \\ &= \bigoplus_{\eta' \in h^{-1}(\eta)} \left( K(X) \otimes_{\mathcal{O}_X} \hat{\mathcal{O}}_{X,\eta'} \right) \\ &= \bigoplus_{\eta^{*'} \in h^{-1}(\eta^*)} \hat{\mathcal{O}}_{X,\eta^{*'}} \end{aligned}$$

which proves (1.13) for  $\eta^*$ .

To complete the proof it will now be enough to consider the case in which  $\eta = (\eta_1, \eta_2)$  for some codimension  $i$  points  $\eta_i$  such that  $\eta_2$  lies on the closure of  $\eta_1$ . By (1.9), we have an isomorphism

$$(1.15) \quad \hat{\mathcal{O}}_{Y,\eta} = \prod_{\alpha} B_{\alpha}$$

where  $\alpha$  runs over the irreducible factors in  $\hat{\mathcal{O}}_{Y,\eta_2}$  of a local equation  $\pi_1$  for  $\eta_1$  in  $\mathcal{O}_{Y,\eta_2}$  (as in 1.7), and  $B_{\alpha}$  is the dvr which is the completion of the local ring of  $\hat{\mathcal{O}}_{Y,\eta}$  at the valuation associated to  $\alpha$ .

We obtain the Parshin chains  $\eta' = (\eta'_1, \eta'_2)$  in  $h^{-1}(\eta)$  by first taking the points  $\eta'_2 \in h^{-1}(\eta_2)$  and by then considering the factorization of  $\pi_1$  in the local ring  $\mathcal{O}_{X,\eta'_2} \supset \mathcal{O}_{Y,\eta_2}$  in order to find the  $\eta'_1$  lying over  $\eta_1$  which contain  $\eta'_2$  in their closure. Here

$$(1.16) \quad \mathcal{O}_X \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{Y,\eta_2} = \bigoplus_{\eta'_2 \in h^{-1}(\eta_2)} \hat{\mathcal{O}}_{X,\eta'_2}.$$

For each irreducible factor  $\alpha$  of  $\pi_1$  in  $\hat{\mathcal{O}}_{Y,\eta_2}$  we consider the factorization of  $\alpha$  into a product of irreducibles in  $\hat{\mathcal{O}}_{X,\eta'_2}$  for  $\eta'_2 \in h^{-1}(\eta_2)$ . These irreducibles give via (1.9) with  $Y$  replaced by  $X$  the dvr summands of each ring  $\hat{\mathcal{O}}_{X,\eta'}$  as  $\eta' = (\eta'_1, \eta'_2)$  runs over the elements of  $h^{-1}(\eta)$ . We see from this that the natural ring homomorphism

$$\mathcal{O}_X \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{Y,\eta} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \left( \prod_{\alpha} B_{\alpha} \right) \rightarrow \bigoplus_{\eta' \in h^{-1}(\eta)} \hat{\mathcal{O}}_{X,\eta'}$$

is the direct sum over  $\alpha$  of the homomorphisms

$$(1.17) \quad \mu_{\alpha} : \mathcal{O}_X \otimes_{\mathcal{O}_Y} B_{\alpha} \rightarrow \bigoplus_{\alpha'} B'_{\alpha'}$$

where  $\alpha'$  ranges over the irreducible factors of  $\alpha$  in  $\hat{\mathcal{O}}_{X,\eta'_2}$  as  $\eta'_2$  ranges over the elements of  $h^{-1}(\eta_2)$ , and where  $B'_{\alpha'}$  is the completion of  $\hat{\mathcal{O}}_{X,\eta'_2}$  with respect to the discrete valuation associated to  $\alpha'$ . To complete the proof of Proposition 1.5 it will suffice to show that (1.17) is an isomorphism.

From (1.16) and the fact that  $X$  is flat and finite over  $Y$  we conclude that the sum

$$(1.18) \quad \bigoplus_{\eta'_2 \in h^{-1}(\eta_2)} \text{Frac}(\hat{\mathcal{O}}_{X,\eta'_2})$$

of the fraction fields of the summands on the right side of (1.16) is an étale algebra of dimension  $[X : Y]$  over the characteristic 0 field  $\text{Frac}(\hat{\mathcal{O}}_{Y,\eta_2})$ . The ring  $B_{\alpha}$  is the completion of the discrete valuation ring of  $\text{Frac}(\hat{\mathcal{O}}_{Y,\eta_2})$  associated to  $\alpha$ . The rings  $B'_{\alpha'}$  on the right side of (1.17) are the completions of the discrete valuation rings of the summands in (1.18) at extensions of the valuation associated to  $\alpha$ . Thus by the theory of discrete valuations in finite separable extensions of fields, we see that the right hand side of (1.17) is a free  $B_{\alpha}$ -module of rank  $[X : Y]$ . The left hand side of (1.17) is a  $B_{\alpha}$ -module which is generated by less than or equal to  $[X : Y]$  elements since  $\mathcal{O}_X$  is a locally free  $\mathcal{O}_Y$ -module of rank  $[X : Y]$ . Thus to show that (1.17) is an isomorphism it will suffice to show that if  $w$  is an element of the residue field  $k(\alpha')$  of a summand  $B'_{\alpha'}$  on the right side of (1.17), then there is an element of the image of  $\mu_{\alpha}$  whose component at  $\alpha'$  is congruent to  $w$  mod the maximal ideal of  $B'_{\alpha'}$  and whose component in any other summand  $B'_{\alpha''}$  appearing on the right in (1.17) is in the maximal ideal of  $B'_{\alpha''}$ .

We know that there is a closed point  $\eta'_2$  lying over  $\eta_2$  such that  $\alpha'$  is an irreducible factor of  $\pi_1$  in  $\hat{\mathcal{O}}_{X,\eta'_2}$  and  $B'_{\alpha'}$  is the completion of the localization of  $\hat{\mathcal{O}}_{X,\eta'_2}$  at the discrete valuation associated to  $\alpha'$ . Thus there are elements  $t, s \in \hat{\mathcal{O}}_{X,\eta'_2}$  such that  $s \notin \hat{\mathcal{O}}_{X,\eta'_2} \cdot \alpha'$  and  $w \equiv t/s$  in  $k(\alpha')$ . By multiplying both  $t$  and  $s$  by the product of a set of representatives for the irreducible factors  $\alpha''$  of  $\pi_1$  in  $\hat{\mathcal{O}}_{X,\eta'_2}$  which are not associate to  $\alpha'$ , we may assume that  $t$  has image in the maximal ideal of  $B'_{\alpha''}$  for all such  $\alpha''$ .

The factorization of  $s$  in the UFD  $\hat{\mathcal{O}}_{X,\eta'_2}$  does not involve  $\alpha'$ , but it might involve some other irreducibles  $\alpha''$  which are irreducible factors of  $\pi_1$ . However, if  $\alpha''$  is such an irreducible, then  $\alpha'' + \alpha'$  is congruent to  $\alpha''$  mod  $\hat{\mathcal{O}}_{X,\eta'_2} \cdot \alpha'$  but not congruent to 0 mod  $\hat{\mathcal{O}}_{X,\eta'_2} \cdot \alpha''$ . We can therefore replace each appearance of an irreducible of the form  $\alpha''$  in the factorization of  $s$  by  $\alpha'' + \alpha'$  so as to be able to assume that  $s \notin \hat{\mathcal{O}}_{X,\eta'_2} \cdot \alpha''$  for all irreducible factors  $\alpha''$  of  $\pi_1$  in  $\hat{\mathcal{O}}_{X,\eta'_2}$  (including  $\alpha'' = \alpha'$ ). Since these  $\alpha''$  define all the discrete valuations of



$\hat{\mathcal{O}}_{X,\eta'_2}$  which lie over the discrete valuation of  $\hat{\mathcal{O}}_{Y,\eta_2}$  associated to  $\alpha$ , we conclude that

$$g = \text{Norm}_{\hat{\mathcal{O}}_{X,\eta'_2}/\hat{\mathcal{O}}_{Y,\eta_2}}(s)$$

is an element of  $\hat{\mathcal{O}}_{Y,\eta_2}$  which does not lie in  $\hat{\mathcal{O}}_{Y,\eta_2} \cdot t_\alpha$ . When we view  $g$  as an element of  $\hat{\mathcal{O}}_{X,\eta'_2}$ , it does not lie in  $\hat{\mathcal{O}}_{X,\eta'_2} \cdot \alpha'$  and it is a multiple of  $s$ . Thus  $w \equiv t/s \equiv t'/g$  in  $k(\alpha')$  where  $t' = t(g/s) \in \hat{\mathcal{O}}_{X,\eta'_2}$ . In view of the isomorphism (1.16), we can now find an element  $q$  of  $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{Y,\eta_2}$  whose image in  $\hat{\mathcal{O}}_{X,\eta'_2}$  is equal to  $t'$  and whose components in  $\hat{\mathcal{O}}_{X,\eta''_2}$  is 0 if  $\eta''_2 \neq \eta'_2$  lies over  $\eta_2$ . It follows that the image of  $q/g$  under the map  $\mu_\alpha$  in (1.17) has the prescribed image  $w$  in the residue field  $k(\alpha')$  of the summand corresponding to  $\alpha'$  and image in the maximal ideal in all the other summands. This completes the proof.  $\square$

**1.c. Adeles and cosimplicial structure.** The construction of the adeles associated to the structure sheaf  $\mathcal{O}_Y$  does not play a major role in this paper. However, we include this subsection since it will pave the way for the crucial construction of the  $K_2$ -adeles associated to  $Y$ .

Recall that  $S(Y)$  is the simplicial set associated to the set  $P(Y)$  of all point of  $Y$  and the order relation defined by  $\eta \geq \eta'$  if  $\eta'$  is a point on the closure  $\bar{\eta}$  of  $\eta$ . The  $n$ -simplex  $S(Y)_n$  is the set of all Parshin  $n+1$ -tuples  $(\eta(0), \dots, \eta(n))$  of points on  $Y$ , these being  $n+1$ -tuples such that  $\eta(0) \geq \eta(1) \geq \dots \geq \eta(n)$ . We define the  $n$ -dimensional adèle group of  $Y$  to be

$$\mathbb{A}'_Y(n) = A(S(Y)_n, \mathcal{O}_Y)$$

in the notation of Definition 1.1. From this definition we see that there is a natural inclusion

$$(1.19) \quad \mathbb{A}'_Y(n) \rightarrow \mathbb{A}_Y(n) = \prod \hat{\mathcal{O}}_{Y,\eta_I}$$

where the product extends over all Parshin  $n+1$ -tuples  $\eta_I := \{\eta_{i_0}, \dots, \eta_{i_n}\}$  on  $Y$ .

Suppose  $I = (i_0, \dots, i_n)$  is an ordered subset of  $J = (j_0, \dots, j_m)$ . From Proposition 1.2 we have a natural map

$$\tau_I^J : \hat{\mathcal{O}}_{Y,\eta_I} \rightarrow \hat{\mathcal{O}}_{Y,\eta_J}.$$

The maps  $\tau_I^J$  may be used to endow the various multicompletions of  $Y$  with a cosimplicial structure.

If we now specify that  $I = (j_0, \dots, \hat{j}_{i_k}, \dots, j_m)$  (so that  $n+1 = m$ ), then we define the coboundary map

$$\mathbb{A}_Y(m-1) \xrightarrow{\partial_{m-1}} \mathbb{A}_Y(m)$$

by stipulating that for  $a \in \hat{\mathcal{O}}_{Y,\eta_I}$ ,  $\partial_{m-1}(a)_J = (-1)^k \tau_I^J(a)$ . This then gives us a complex

$$\mathbb{A}_Y^\bullet : \quad \mathbb{A}_Y(0) \xrightarrow{\partial_0} \mathbb{A}_Y(1) \xrightarrow{\partial_1} \dots \xrightarrow{\partial_{d-1}} \mathbb{A}_Y(d)$$

when  $d = \dim(Y)$ . There are degeneracy maps induced by mapping the Parshin cycle  $(\eta_{i_0}, \dots, \eta_{i_{m-1}})$  of length  $m$  to the Parshin cycle  $(\eta_{i_0}, \dots, \eta_{i_k}, \eta_{i_k}, \dots, \eta_{i_{m-1}})$  of length  $m+1$ . By [27, §2], the inclusion (1.19) gives a complex

$$(1.20) \quad \mathbb{A}_Y'^\bullet : \quad \mathbb{A}_Y'(0) \xrightarrow{\partial_0} \mathbb{A}_Y'(1) \xrightarrow{\partial_1} \dots \xrightarrow{\partial_{d-1}} \mathbb{A}_Y'(d)$$

which we will call the adelic complex of  $Y$  and which has degeneracy maps defined in the above way.

When there is no confusion, we will also use the symbols  $\mathbb{A}_Y^\bullet$  and  $\mathbb{A}'_Y$  to denote the complexes defined as above but using only non-degenerate Parshin cycles  $(\eta_{i_0}, \dots, \eta_{i_{m-1}})$  in which the  $\eta_{i_j}$  are all distinct. Omitting such degenerate cycles does not effect the cohomology of the complexes we consider – see the remark after (1) on page 179 of [42].

1.c.1. We conclude this section by considering the case in which  $Y$  is a regular integral scheme of dimension 2. We will recall from [42] the local conditions on elements  $\mathbb{A}_Y(2)$  which are necessary and sufficient for these elements to lie in  $\mathbb{A}'_Y(2)$ . This motivates the definition of  $K_2$ -adeles to be given in the next section.

Recall that  $K(Y)$  denotes the function field of  $Y$ , so that  $K(Y)$  may be identified with the two sheaves of rings on a point  $\mathcal{O}_{Y,\eta_0}$  and  $\hat{\mathcal{O}}_{Y,\eta_0}$ . We start by considering a Parshin triple  $(\eta_0, \eta_1, \eta_2)$  on  $Y$  and we recall that  $\hat{\mathcal{O}}_{Y,\eta_0\eta_2} = K(Y) \cdot \hat{\mathcal{O}}_{Y,\eta_2}$ . We write  $\hat{\mathcal{O}}_{Y,\eta_2}[\eta_1^{-1}]$  for the subring of elements in  $\hat{\mathcal{O}}_{Y,\eta_0\eta_2}$  which are regular off the curve  $\bar{\eta}_1$  and denote by  $v_{\eta_1}$  the valuation of  $K(Y)$  that corresponds to  $\bar{\eta}_1$ .

We let  $v_{\eta_1\eta_2}$  denote a discrete valuation on  $\hat{\mathcal{O}}_{Y,\eta_1\eta_2}$  corresponding to one of the components (branches) as in Lemma 1.4 and let  $\mathfrak{p}_{\eta_1\eta_2}$  denote the corresponding prime ideal of  $\hat{\mathcal{O}}_{Y,\eta_1\eta_2}$ . We then identify  $\mathbb{A}'_Y(2)$  with the restricted direct product

$$\mathbb{A}'_Y(2) = \prod'_{(\eta_0, \eta_1, \eta_2)} \hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2} \subset \mathbb{A}_Y(2) = \prod_{(\eta_0, \eta_1, \eta_2)} \hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}$$

consisting of all elements of  $\mathbb{A}_Y(2)$  whose terms  $(f_{\eta_0\eta_1\eta_2})$  with

$$f_{\eta_0\eta_1\eta_2} \in \hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2} = \text{Frac}(\hat{\mathcal{O}}_{Y,\eta_1\eta_2})$$

satisfy the following two properties (cf. page 179 in [42]):

P1. (Adelic Property 1) There exists a divisor  $D$  on  $Y$  such that for each codimension one point  $\eta_1$  on  $Y$ , each  $\eta_2 \leq \eta_1$  (and each branch of  $\bar{\eta}_1$  at  $\eta_2$ ) we have

$$v_{\eta_1\eta_2}(f_{\eta_0\eta_1\eta_2}) \geq v_{\eta_1}(D);$$

P2. (Adelic Property 2) Suppose that  $\eta_1$  is a codimension one point on  $Y$ . Then for any positive integer  $k$ , for all but a finite number of  $\eta_2$  on  $\bar{\eta}_1$ , we have

$$f_{\eta_0\eta_1\eta_2} \in \hat{\mathcal{O}}_{Y,\eta_2}[\eta_1^{-1}] + \mathfrak{p}_{\eta_1\eta_2}^k.$$

## 2. EQUIVARIANT ADELIC CHOW GROUPS

2.a. **Generalities.** For  $\ell \geq 0$  and for a ring  $S$  we let  $K_\ell(S)$  denote the  $\ell$ -th K-group of the ring  $S$ . For a two-sided ideal  $\mathcal{A}$  of  $S$  we set  $\bar{S} = S/\mathcal{A}$  for the quotient ring. Recall from [32, Theorem 6.2] that we have the long exact sequence

$$(2.1) \quad \begin{aligned} K_2(S, \mathcal{A}) \rightarrow K_2(S) \rightarrow K_2(\bar{S}) \rightarrow K_1(S, \mathcal{A}) \rightarrow K_1(S) \rightarrow \\ \rightarrow K_1(\bar{S}) \rightarrow K_0(S, \mathcal{A}) \rightarrow K_0(S) \rightarrow K_0(\bar{S}). \end{aligned}$$

Recall that  $K_1(S, \mathcal{A})$  may be described as the quotient group

$$(2.2) \quad K_1(S, \mathcal{A}) = \frac{\mathrm{GL}(S, \mathcal{A})}{\mathrm{E}(S, \mathcal{A})}$$

where  $\mathrm{GL}(S, \mathcal{A})$  is the subgroup of elements in the full general linear group  $\mathrm{GL}(S)$  which are congruent to the identity mod  $\mathcal{A}$  and  $\mathrm{E}(S, \mathcal{A})$  is the smallest normal subgroup of  $\mathrm{GL}(S)$  containing the elementary matrices  $e_{ij}(a)$  for all  $a \in \mathcal{A}$ . (See for instance page 93 in [46].)

**2.b.  $K_\ell$ -adeles of arithmetic surfaces.** We suppose in this section that  $Y$  is an irreducible regular flat projective scheme over  $\mathbb{Z}$  and that  $\ell$  is either 1 or 2. We now make the following important definitions (cf. Definition 10 on page 719 of [39]):

**Definition 2.1.** a) We define

$$K_\ell(\mathbb{A}_{Y,012}[G]) = \prod_{\eta_0 \eta_1 \eta_2} K_\ell(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}[G])$$

where the product is over all non-degenerate Parshin triples.

- b) For  $0 \leq i < j \leq 2$  we define  $K_\ell(\mathbb{A}_{Y,ij}[G]) = \prod_{\eta_i \eta_j} K_\ell(\hat{\mathcal{O}}_{Y, \eta_i \eta_j}[G])$  where the product is over all non-degenerate Parshin pairs consisting of a codimension  $i$  point  $\eta_i$  and a codimension  $j$  point  $\eta_j < \eta_i$ .
- b) For  $0 \leq i \leq 2$  we define  $K_\ell(\mathbb{A}_{Y,i}[G]) = \prod_{\eta_i} K_\ell(\hat{\mathcal{O}}_{Y, \eta_i}[G])$  where the product is over all points  $\eta_i$  of codimension  $i$ .

**Definition 2.2.** a) We define  $K'_\ell(\mathbb{A}_{Y,012}[G])$  to be the restricted product

$$K'_\ell(\mathbb{A}_{Y,012}[G]) = \prod' K_\ell(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}[G])$$

consisting of elements  $(\kappa_{\eta_0 \eta_1 \eta_2})$  as  $(\eta_0, \eta_1, \eta_2)$  ranges over all non-degenerate Parshin triples for which  $\kappa_{\eta_0 \eta_1 \eta_2} \in K_\ell(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}[G])$  satisfies the following two properties:

- (PK1) Almost all  $\eta_1$  have the property that  $\kappa_{\eta_0 \eta_1 \eta_2} \in K_\ell(\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}[G])^\flat$  for all  $\eta_2 < \eta_1$ , where  $K_\ell(\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}[G])^\flat$  denotes the image of  $K_\ell(\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}[G])$  in  $K_\ell(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}[G])$ .
- (PK2) Given  $\eta_1$  and a positive integer  $k$  then for all but a finite number of closed points  $\eta_2$  on  $\bar{\eta}_1$

$$\kappa_{\eta_0 \eta_1 \eta_2} \in K_\ell(\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}[G], \mathfrak{p}_{\eta_1 \eta_2}^k)^\flat \cdot K_\ell(\hat{\mathcal{O}}_{Y, \eta_2}[\eta_1^{-1}][G])^\flat$$

where  $\hat{\mathcal{O}}_{Y, \eta_2}[\eta_1^{-1}]$  denotes the subring of elements in  $\hat{\mathcal{O}}_{Y, \eta_0 \eta_2}$  which are regular off the curve  $\bar{\eta}_1$ .

(Note that these properties parallel the restricted direct product conditions (P1) and (P2) at the end of §1.c. )

- b1) We define  $K'_\ell(\mathbb{A}_{Y,01}[G])$  to be the subgroup of elements  $(\kappa_{\eta_0 \eta_1}) \in \prod_{\eta_1} K_\ell(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1}[G])$  with the property that  $\kappa_{\eta_0 \eta_1} \in K_\ell(\hat{\mathcal{O}}_{Y, \eta_1}[G])^\flat$  for almost all  $\eta_1$ .
- b2) We define  $K'_\ell(\mathbb{A}_{Y,12}[G]) = K_\ell(\mathbb{A}_{Y,12}[G]) = \prod_{\eta_2} K_\ell(\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}[G])$ , i.e. we impose no restriction in this case.

- b3) We define  $K'_\ell(\mathbb{A}_{Y,02}[G])$  to be the subgroup of  $K_\ell(\mathbb{A}_{Y,02}[G]) = \prod_{\eta_2} K_\ell(\hat{\mathcal{O}}_{Y,\eta_0\eta_2}[G])$  consisting of  $x = (x_{\eta_0\eta_2})_{\eta_2}$  with the following property: There is a divisor  $D \subset Y$  (that could depend on  $x$ ) such that: For all  $\eta_2$ ,  $x_{\eta_0\eta_2}$  is in  $K_\ell(\hat{\mathcal{O}}_{Y,\eta_2}[D^{-1}][G])^b$  where  $\hat{\mathcal{O}}_{Y,\eta_2}[D^{-1}]$  is the subring of  $\hat{\mathcal{O}}_{Y,\eta_0\eta_2}$  consisting of elements which are regular off  $D$ .
- c) We define  $K'_\ell(\mathbb{A}_{Y,i}[G]) = K_\ell(\mathbb{A}_{Y,i}[G]) = \prod_{\eta_i} K_\ell(\hat{\mathcal{O}}_{Y,\eta_i}[G])$ , i.e. we impose no restriction.

**Remark 2.3.** The group  $K'_\ell(\mathbb{A}_{Y,ij}[G])$  maps diagonally to  $K_\ell(\mathbb{A}_{Y,012}[G])$ . We can see that the image  $K'_\ell(\mathbb{A}_{Y,02}[G])^b$  is actually a subgroup of the restricted product  $K'_\ell(\mathbb{A}_{Y,012}[G])$ . This is not necessarily true for the images of  $K'_\ell(\mathbb{A}_{Y,01}[G])$  and  $K'_\ell(\mathbb{A}_{Y,12}[G])$ .

## 2.c. The adelic Chow groups.

**Definition 2.4.** For  $\ell \in \{1, 2\}$ , the  $\ell$ -th equivariant adelic Chow group is defined to be

$$(2.3) \quad \text{CH}_\mathbb{A}^1(Y[G]) = \frac{K'_1(\mathbb{A}_{Y,01}[G])}{\prod_{0 \leq i \leq 1} K_1(\mathbb{A}_{Y,i}[G])^b},$$

$$(2.4) \quad \text{CH}_\mathbb{A}^2(Y[G]) = \frac{K'_2(\mathbb{A}_{Y,012}[G]) \cdot \prod_{0 \leq i < j \leq 2} K'_2(\mathbb{A}_{Y,ij}[G])^b}{\prod_{0 \leq i < j \leq 2} K_2(\mathbb{A}_{Y,ij}[G])^b}$$

where again, the superscript  $b$  denotes the image of the corresponding group in the unrestricted product  $\prod_{\eta_0\eta_1} K_1(\hat{\mathcal{O}}_{Y,\eta_0\eta_1}[G])$ , resp.  $\prod_{\eta_0\eta_1\eta_2} K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}[G])$ .

**Remark 2.5.** a) In the case of  $\text{CH}_\mathbb{A}^1(Y[G])$ , both numerator and denominator are subgroups of  $\prod_{\eta_0\eta_1} K_1(\hat{\mathcal{O}}_{Y,\eta_0\eta_1}[G])$ . In the case of  $\text{CH}_\mathbb{A}^2(Y[G])$ , they are both subgroups of  $\prod_{\eta_0\eta_1\eta_2} K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}[G])$ .

b) If  $G = \{1\}$  and  $Y$  is a smooth algebraic surface over a field, we have an isomorphism  $\text{CH}_\mathbb{A}^1(Y) \xrightarrow{\sim} \text{Pic}(Y)$ . We can also see using [39] that, in this case, there is a natural group homomorphism  $\text{CH}^2(Y) \rightarrow \text{CH}_\mathbb{A}^2(Y)$ . Here  $\text{CH}^2(Y)$  denotes the classical Chow group of algebraic cycles of codimension 2 on  $Y$ . However, it is not clear that this is an isomorphism since the definition of the adelic Chow group in [39] is somewhat different. Nevertheless, we conjecture that this is the case.

2.c.1. Here we recall Fröhlich's adelic description of the class group of a group ring; for details see [49] and [17]. We define  $\text{Cl}(\mathbb{Z}[G])$  to be the kernel of the extension of scalars map  $\ker(K_0(\mathbb{Z}[G]) \rightarrow K_0(\mathbb{Q}[G]))$ . By [48], this coincides with the subgroup  $K_0^{\text{red}}(\mathbb{Z}[G])$  of  $K_0(\mathbb{Z}[G])$  generated by elements of the form  $[M] - \text{rank}(M) \cdot [\mathbb{Z}[G]]$ . Then from Ch. I Sect. 3 in [49] and Ch. II Sect. 1 in [17] we know that, there is a natural isomorphism

$$(2.5) \quad \text{Cl}(\mathbb{Z}[G]) \cong \frac{\prod'_p K_1(\mathbb{Q}_p[G])}{K_1(\mathbb{Q}[G])^b \prod_p K_1(\mathbb{Z}_p[G])^b}.$$

Here:  $K_1(\mathbb{Z}_p[G])^b$  denotes the image of  $K_1(\mathbb{Z}_p[G])$  in  $K_1(\mathbb{Q}_p[G])$ ; the restricted product  $\prod'_p K_1(\mathbb{Q}_p[G])$  in the numerator consists of elements almost all of whose terms lie in the subgroup  $K_1(\mathbb{Z}_p[G])^b$ ; and  $K_1(\mathbb{Q}[G])^b$  denotes the image of  $K_1(\mathbb{Q}[G])$  in  $\prod'_p K_1(\mathbb{Q}_p[G])$ . Now notice that we can interpret the right hand side of (2.5) as  $\text{CH}_\mathbb{A}^1(\text{Spec}(\mathbb{Z})[G])$  (see Definition 2.4). Hence, we obtain an isomorphism  $\text{Cl}(\mathbb{Z}[G]) \cong \text{CH}_\mathbb{A}^1(\text{Spec}(\mathbb{Z})[G])$ . See also §5.3.

## 2.d. $\mathrm{SK}_1$ of $p$ -adic group rings.

2.d.1. Throughout this subsection  $R$  will always denote a commutative ring which is an integral domain with field of fractions  $N$  of characteristic zero, and  $N^c$  will denote a chosen algebraic closure of  $N$ . We define the group  $\mathrm{SK}_1(R)$  to be the kernel of the group homomorphism  $\mathrm{Det} : \mathrm{K}_1(R) \rightarrow \mathrm{K}_1(N^c) = (N^c)^\times$  induced by ring extension. We recall from [12, 45.12, p. 142] that if  $R$  is in addition local, then  $\mathrm{SK}_1(R) = \{1\}$ .

**Lemma 2.6.** *For any field  $N$  of characteristic zero and for an indeterminate  $t$  we have*

$$\mathrm{SK}_1(N[t, t^{-1}]) = \{1\}.$$

*Proof.* In §9.b.2 we show that  $\mathrm{Det}$  is injective on  $\mathrm{K}_1(N[t, t^{-1}])$ . □

We also refer to [4, Prop., p. 354] for the proof of the following:

**Proposition 2.7.** (Bloch) *If  $R$  is a local Noetherian domain and  $f \in R$  is such that the localization  $R_f$  is regular, then  $\mathrm{SK}_1(R_f) = (0)$ .*

**Corollary 2.8.** *If  $R_S$  is the localization of a local Noetherian regular domain  $R$  at an arbitrary multiplicatively closed subset  $S$  of  $R - \{0\}$ , then we have  $\mathrm{SK}_1(R_S) = (0)$ .*

2.d.2. We now consider the case of group rings and we again denote by  $\mathrm{Det}$  the map

$$(2.6) \quad \mathrm{Det} : \mathrm{K}_1(R[G]) \rightarrow \mathrm{K}_1(N^c[G]) = \bigoplus_\chi (N^c)^\times$$

where the direct sum extends over the irreducible  $N^c$ -valued characters  $\chi$  of  $G$ . We write  $\mathrm{SK}_1(R[G]) = \ker(\mathrm{Det})$ , so that we have the exact sequence

$$(2.7) \quad 1 \rightarrow \mathrm{SK}_1(R[G]) \rightarrow \mathrm{K}_1(R[G]) \rightarrow \mathrm{Det}(\mathrm{K}_1(R[G])) \rightarrow 1.$$

We also define  $\mathrm{SL}(R[G])$  to be the kernel of the composite homomorphism

$$(2.8) \quad \mathrm{SL}(R[G]) = \ker \left( \mathrm{GL}(R[G]) \rightarrow \mathrm{K}_1(R[G]) \xrightarrow{\mathrm{Det}} \mathrm{K}_1(N^c[G]) \right).$$

Clearly  $\mathrm{E}(R[G]) \subset \mathrm{SL}(R[G])$  and we have the equality  $\mathrm{E}(R[G]) = \mathrm{SL}(R[G])$  precisely when  $\mathrm{SK}_1(R[G]) = (1)$ . Recall that if  $R$  is the ring of integers of a  $p$ -adic field, then  $\mathrm{SK}_1(R[G])$  is completely described in Oliver's papers, see [38].

2.d.3. Suppose now in addition that  $R$  is a dvr with maximal ideal  $\mathfrak{p}$  and uniformizer  $\pi$ . Let  $\hat{N}$ , resp.  $\hat{R}$ , denote the  $\mathfrak{p}$ -adic completion of the fraction field  $N$ , resp.  $R$ . We denote by  $\mathrm{SL}(\hat{R}[G], \mathfrak{p}^m)$  the subgroup of  $\mathrm{SL}(\hat{R}[G])$  consisting of matrices which are congruent to the identity modulo  $\mathfrak{p}^m$ .

Recall that we say that the group algebra  $N[G]$  splits if we can write

$$(2.9) \quad N[G] = \prod_i M_{m_i}(Z_i),$$

where each  $Z_i$  is a commutative finite field extension of  $N$ .

**Lemma 2.9.** *Assume  $N[G]$  splits as above. For  $m \geq 0$  we have*

- (a)  $\mathrm{SL}(\hat{N}[G]) = \mathrm{SL}(N[G]) \cdot \mathrm{SL}(\hat{R}[G], \mathfrak{p}^m)$ ;
- (b)  $\mathrm{SL}(\hat{R}[G]) = \mathrm{SL}(R[G]) \cdot \mathrm{SL}(\hat{R}[G], \mathfrak{p}^m)$ .

*Proof.* We prove (a), and note that (b) follows easily from (a). We let  $\hat{\mathfrak{M}}_{R,G}$  denote a maximal  $\hat{R}$ -order in  $\hat{N}[G]$ . Clearly we can take  $\hat{\mathfrak{M}}_{R,G} = \hat{R}[G]$  unless the residue characteristic of  $R$  divides the order of  $G$ . Under our assumption on  $N[G]$  above we can take

$$\hat{\mathfrak{M}}_{R,G} = \prod_i M_{m_i}(\hat{\mathcal{O}}_{Z_i}).$$

Write also  $M_n(\hat{N}[G]) = \prod_i M_{n_i}(\hat{Z}_i)$ . We choose  $r$  such that  $\pi^r \hat{\mathfrak{M}}_{R,G} \subset \hat{R}[G]$  and we set  $a = r + m$ . Note that, as  $N$  is dense in  $\hat{N}$ , we know that for any non-negative integer  $a$  we have the equality

$$\mathrm{GL}_n(\hat{N}[G]) = \mathrm{GL}_n(N[G]) \cdot \mathrm{GL}(\hat{R}[G], \mathfrak{p}^a).$$

Let  $\hat{x} \in \mathrm{SL}_n(\hat{N}[G])$  and choose  $y \in \mathrm{GL}_n(N[G])$  close to  $\hat{x}$ , so that  $\hat{x}y^{-1} = 1 + \pi^a \lambda$  with  $\lambda \in M_n(\hat{R}[G])$ . Then

$$\mathrm{Det}(1 + \pi^a \lambda) = \mathrm{Det}(y)^{-1} \in \mathrm{Det}(\mathrm{GL}_n(N[G])) \cap \mathrm{Det}(1 + \pi^a M_n(\hat{R}[G])).$$

We write  $1 + \pi^a \lambda = \prod_i 1 + \pi^a \lambda_i$  with  $\lambda_i \in M_{n_i}(\hat{\mathcal{O}}_{Z_i})$ . As  $N[G]$  is semi-local we can write  $y = \prod_i y_i = \prod_i e_i \delta_i d_i$  where the  $e_i$  and  $d_i$  lie in the group of elementary matrices  $E(Z_i)$ , and where  $\delta_i$  is diagonal matrix with all non-leading terms 1; so that the leading diagonal term  $\xi_i$  must have  $\det(y_i) = \xi_i \in Z_i^\times$ . By Lemma 2.2.b in [11] we have a similar decomposition

$$1 + \pi^a \lambda = \prod_i 1 + \pi^a \lambda_i = \prod_i e'_i \delta'_i d'_i$$

where the  $e'_i$  and  $d'_i$  lie in the group of elementary matrices  $E(\mathcal{O}_{Z_i}, \mathfrak{p}^a)$ , and where  $\delta'_i$  is diagonal with all non-leading terms 1; so that the leading diagonal term must be  $\xi'_i$  with

$$\det(y_i^{-1}) = \det(1 + \pi^a \lambda_i) = \xi'_i \in 1 + \pi^a \hat{\mathcal{O}}_{Z_i}.$$

Thus we have shown that

$$\xi_i^{-1} = \det(y_i^{-1}) = \det(1 + \pi^a \lambda_i) = \xi'_i \in Z_i \cap (1 + \pi^a \hat{\mathcal{O}}_{Z_i}) = 1 + \pi^a \mathcal{O}_{Z_i}.$$

We set  $\delta = \prod_i \delta_i$ ; we can then write

$$\hat{x} = (y\delta^{-1}) \cdot \delta(1 + \pi^a \lambda) \in \mathrm{SL}_n(N[G]) \cdot \mathrm{SL}_n(\hat{R}[G], \mathfrak{p}^m)$$

since  $y\delta^{-1} \in \mathrm{SL}_n(N[G])$  and  $\delta(1 + \pi^a \lambda) \in 1 + \pi^a \hat{\mathfrak{M}}_{R,G} \subset 1 + \pi^m \hat{R}[G]$ .  $\square$

2.d.4. In this paragraph, we recall some results from [9] (see the introduction of loc. cit.).

**Theorem 2.10.** *Suppose that  $R$  is a Noetherian domain with fraction field of characteristic zero. Assume that the natural map  $R \rightarrow \varprojlim_n R/p^n R$  is an isomorphism, so that  $R$  is  $p$ -adically complete. Then for any integer  $k \geq 2$ ,  $K_1(R[G], (p)^k)$  is a subgroup of  $K_1(R[G])$  and we have*

$$K_1(R[G], (p)^k) \cap \mathrm{SK}_1(R[G]) = \{1\}.$$

*Proof.* This follows from [9] Theorems 1.3 and 1.4.  $\square$

**Corollary 2.11.** *Let  $R$  be a discrete valuation ring of mixed characteristic with fraction field  $N$  and denote by  $\hat{R}$  its  $p$ -adic completion. Assume that  $N[G]$  splits as in (2.9). Then the natural map  $\mathrm{SK}_1(R[G]) \rightarrow \mathrm{SK}_1(\hat{R}[G])$  is surjective.*



*Proof.* Define  $\mathrm{SK}_1(R/(p)^m[G])$  to be the image of  $\mathrm{SK}_1(R[G])$  in  $K_1(R/(p)^m[G])$ . Theorem 2.10 implies that, for  $m \geq 2$ , the map

$$\mathrm{SK}_1(\hat{R}[G]) \rightarrow \mathrm{SK}_1(\hat{R}/(p)^m[G])$$

is injective and hence an isomorphism. The result now follows from Lemma 2.9 (b).  $\square$

In [9] we obtain more precise results about  $\mathrm{SK}_1$  when we assume that, among other additional hypotheses, our coefficient rings afford a lift of Frobenius. We are going to use the following corollaries of the main result of [9]. (Since carefully stating this main result would require a fair amount of additional explanation, we prefer not to do this at this time.) Here we will assume that  $W = W(k)$  is the ring of integers in a finite unramified extension of  $\mathbb{Q}_p$  with residue field  $k$ .

**Corollary 2.12.** *Suppose that  $W$  is as above and  $t$  an indeterminate. Then the inclusion  $W \subset W[[t]]$  induces an isomorphism  $\mathrm{SK}_1(W[G]) \xrightarrow{\sim} \mathrm{SK}_1(W[[t]][G])$ .*

Recall that if  $R$  is the ring of integers of the finite extension  $N$  of  $\mathbb{Q}_p$ , we denote by  $R\langle\langle t \rangle\rangle$  the  $p$ -adic completion of the polynomial ring  $R[t]$  and by  $R\{\{t\}\}$  the  $p$ -adic completion of the Laurent power series ring  $R((t)) = R[[t]][t^{-1}]$ . Then  $N \otimes_R R\langle\langle t \rangle\rangle$  is the free Tate algebra  $N\{t\}$  in one variable over  $N$ .

**Corollary 2.13.** *Suppose that  $W$  is as above and  $t$  an indeterminate.*

*a) The inclusion  $W\langle\langle t^{-1} \rangle\rangle \subset W\{\{t\}\}$  induces an isomorphism*

$$\mathrm{SK}_1(W\langle\langle t^{-1} \rangle\rangle[G]) \xrightarrow{\sim} \mathrm{SK}_1(W\{\{t\}\}[G]).$$

*b) The inclusion  $W[[t]] \subset W\{\{t\}\}$  induces an injection*

$$\mathrm{SK}_1(W[[t]][G]) \hookrightarrow \mathrm{SK}_1(W\{\{t\}\}[G]).$$

2.d.5. Let us also record:

**Lemma 2.14.** *Suppose that  $\mathbb{Q}_p[G]$  splits. Then we have:*

*a)  $\mathrm{SK}_1(\mathbb{Q} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p\langle\langle t^{-1} \rangle\rangle[G]) = \{1\}$ ,*

*b) If also  $p$  does not divide the order of  $G$ , we have  $\mathrm{SK}_1(\mathbb{Z}_p\langle\langle t^{-1} \rangle\rangle[G]) = \{1\}$ .*

*Proof.* By Morita equivalence, it is enough to show that  $\mathrm{SK}_1(R\langle\langle t^{-1} \rangle\rangle) = (1)$  and  $\mathrm{SK}_1(N \otimes_R R\langle\langle t^{-1} \rangle\rangle) = \mathrm{SK}_1(N\{t^{-1}\}) = (1)$ , where  $R$  are the integers in a finite extension  $N$  of  $\mathbb{Q}_p$  and  $N\{t\}$  is a Tate algebra. This first statement follows from [23, proof of IV, Prop. 4] applied to  $A = R$ ,  $B = R\langle\langle t^{-1} \rangle\rangle$ : Indeed, Gruson's argument implies that the natural map  $\mathrm{SK}_1(R[t^{-1}]) \rightarrow \mathrm{SK}_1(B)$  is surjective and the result follows since  $\mathrm{SK}_1(R[t^{-1}]) = (1)$ . The proof of  $\mathrm{SK}_1(N\{t^{-1}\}) = (1)$  is similar. In fact, this is a special case of [23, Theorem 1].  $\square$

### 3. LATTICES, DETERMINANT FUNCTORS AND DETERMINANT THEORIES

In what follows,  $R$  is a commutative Noetherian ring,  $A$  is a commutative Noetherian flat  $R$ -algebra and  $t$  a non-zero divisor in  $A$  such that  $A/tA$  is finitely generated and flat over  $R$ . We also consider  $A_t = A[t^{-1}]$ . In the main examples we have in mind,  $A = R[t]$ , or  $A = R[[t]]$ . Also all modules over a group ring such as  $A[G]$  are left modules.

3.a. **Some lemmas.** We start with:

**Lemma 3.1.** *Suppose that  $S$  is a local Noetherian commutative ring with 1 and residue field  $k$  of characteristic  $p$ . Suppose that  $P$  is  $p$ -Sylow subgroup of  $G$ . If  $p = 0$ , take  $P = \{1\}$ . Let  $M$  be a finitely generated  $S[G]$ -module. Then  $M$  is  $S[G]$ -projective if and only if the  $S[P]$ -module  $M$  obtained by restriction of operators from  $G$  to  $P$  is  $S[P]$ -projective.*

*Proof.* Observe that since  $[G : P]$  is invertible in  $S$ , the  $S[G]$ -module  $S[G/P]$  admits the  $S[G]$ -module  $S$  with trivial  $G$ -action as a direct summand. By Frobenius reciprocity

$$S[G/P] \otimes_S M \simeq S[G] \otimes_{S[P]} (\text{Res}_{G \rightarrow P}(M)).$$

Hence,  $M$  is a direct summand of  $S[G] \otimes_{S[P]} (\text{Res}_{G \rightarrow P}(M))$ . The result follows from this.  $\square$

**Lemma 3.2.** *Suppose that  $S$  is a local Noetherian commutative ring with residue field  $k$  of characteristic  $p$ . Suppose that  $G$  is a  $p$ -group. ( $G = \{1\}$ , if  $p = 0$ .) Let  $M$  be a finitely generated  $S[G]$ -module. Let  $J$  be the Jacobson radical of  $S[G]$ . Then the following are equivalent:*

- a)  $M$  is  $S[G]$ -free,
- b)  $M$  is  $S[G]$ -projective,
- c)  $M$  is  $S[G]$ -flat,
- d)  $\text{Tor}_1^{S[G]}(S[G]/J, M) = (0)$ .

*Proof.* Notice that since  $S$  is Noetherian,  $S[G]$  is also Noetherian. Clearly (a) implies (b), (b) implies (c), (c) implies (d). It remains to show that (d) implies (a). Recall,  $G$  is a  $p$ -group. In this case,  $S[G]/J = k$ . Suppose that  $\phi : k^n \xrightarrow{\sim} M/JM$ . Lift  $\phi$  to an  $S[G]$ -module homomorphism

$$0 \rightarrow K \rightarrow S[G]^n \xrightarrow{\Phi} M \rightarrow 0$$

with  $K$  the (finitely generated) kernel. By the non-commutative version of Nakayama's lemma,  $\Phi$  is surjective. By tensoring the exact sequence above with  $S[G]/J \otimes_{S[G]} -$  we obtain (using (d)) an exact sequence of  $S[G]/J$ -modules

$$0 \rightarrow K/JK \rightarrow (S[G]/J)^n \xrightarrow{\phi} M/JM \rightarrow 0.$$

But  $\phi$  is an isomorphism so  $K/JK = (0)$ . Another application of Nakayama's lemma now gives  $K = (0)$  and so  $M$  is actually free.  $\square$

3.b. **Lattices.** Suppose  $M_0$  is a finitely generated projective  $R[G]$ -module. We set  $\mathcal{M} = M_0 \otimes_R A_t$  and  $L_0 = M_0 \otimes_R A$ .

**Definition 3.3.** *A finitely generated projective  $A[G]$ -submodule  $L$  of  $\mathcal{M} = M_0 \otimes_R A_t$  with  $\sum_{n \leq 0} L \cdot t^n = \mathcal{M}$ , will be called an “ $A[G]$ -lattice”, or simply a “lattice”.*

Notice that for a lattice  $L$  there is  $n \geq 0$  such that  $t^n L_0 \subset L \subset t^{-n} L_0$  and we have a canonical  $A_t[G]$ -isomorphism  $L \otimes_A A_t = \mathcal{M}$ .

**Proposition 3.4.** *Suppose that  $L \subset \mathcal{M} = M_0 \otimes_R A_t$  is a finitely generated  $A[G]$ -submodule of  $\mathcal{M}$ . Then  $L$  is a lattice if and only if the following condition is satisfied: There is  $n \geq 0$ , such that  $t^n L_0 \subset L \subset t^{-n} L_0$ , and the quotients  $L/t^n L_0$ ,  $t^{-n} L_0/L$  are both  $R[G]$ -projective.*

*Proof.* First assume that  $L$  is  $A[G]$ -projective. Consider the exact sequence

$$(3.1) \quad 0 \rightarrow t^n L_0 \rightarrow L \rightarrow L/t^n L_0 \rightarrow 0.$$

To show that  $L/t^n L_0$ ,  $t^{-n} L_0/L$ , are  $R[G]$ -projective it is enough to reduce to the case that  $R$  is local Noetherian with residue field of characteristic  $p$  and by Lemma 3.1 suppose that  $G$  is a  $p$ -group. Set  $\bar{L} = L/t^n L_0$ . Since  $A$  is flat over  $R$ , the  $R[G]$ -modules  $L$  and  $t^n L_0$  are also flat. The exact sequence (3.1) implies that  $\bar{L}$  has  $R[G]$ -Tor dimension  $\leq 1$ . Since  $\bar{L}$  is finitely generated over  $R[G]$ , Lemma 3.2 above and a standard argument shows that  $\bar{L}$  has  $R[G]$ -projective dimension  $\leq 1$ . The same is true for the  $R[G]$ -module  $t^{-n} L/L_0$ . Assume that the  $R[G]$ -projective dimension of  $t^{-n} L_0/L$  is 1. The exact sequence

$$0 \rightarrow \bar{L} \rightarrow t^{-n} L_0/t^n L_0 \rightarrow t^{-n} L_0/L \rightarrow 0$$

would then give that the projective dimension of  $t^{-n} L/L_0$  is  $> 1$ , a contradiction. Hence,  $\bar{L}$  is  $R[G]$ -projective. The same argument now shows that  $t^{-n} L_0/L_0$  is also  $R[G]$ -projective. We conclude that  $L/t^n L_0$ ,  $t^{-n} L_0/L$  are both  $R[G]$ -projective.

Conversely, assume

$$t^n L_0 \subset L \subset t^{-n} L_0$$

and that the quotients  $L/t^n L_0$ ,  $t^{-n} L_0/L$  are both  $R[G]$ -projective. We will show that  $L$  is  $A[G]$ -projective.

We can assume that  $A$  and  $R$  are local and that  $t$  is in the unique maximal ideal of  $A$ . (Otherwise,  $t$  is invertible and we get  $L = L_0$  in the corresponding localization.) In addition, by Lemma 3.1, we can suppose that  $G$  is a  $p$ -group where  $p$  is the characteristic of the residue field of  $A$ . We first claim that it is enough to show that, under our assumptions,  $L/tL$  is  $R[G]$ -free. Indeed, we will first show that if  $L/tL$  is  $R[G]$ -free, then  $L$  is  $A[G]$ -free. Consider a map  $F \rightarrow L$  from a free  $A[G]$ -module which lifts  $F/tF \xrightarrow{\sim} L/tL$ . By Nakayama's lemma,  $F \rightarrow L$  is surjective; let  $K$  be its kernel. Now notice that since  $L \subset \mathcal{M}$ ,  $L$  is  $t$ -torsion free and so

$$0 \rightarrow K/tK \rightarrow F/tF \rightarrow L/tL \rightarrow 0$$

is exact. Hence,  $K/tK = (0)$ . Since  $t$  is in the maximal ideal of the local ring  $A$ , by Nakayama's lemma again,  $K = (0)$ . It now remains to show that  $\bar{L} := L/tL$  is  $R[G]$ -free. For simplicity, set  $L_n = t^{-n} L_0$  which is  $A[G]$ -free. By our assumption and Lemma 3.2,  $L_n/L$  is  $R[G]$ -free. By enlarging  $n$  if needed, we can assume that  $L \subset tL_n$ . Now tensor the exact  $A$ -sequence

$$0 \rightarrow L \rightarrow L_n \rightarrow L_n/L \rightarrow 0$$

with  $- \otimes_A A/tA$ . Since  $t$  is not a zero-divisor in  $A$ , we obtain

$$0 \rightarrow T(L_n/L) \rightarrow L/tL \rightarrow L_n/tL_n \rightarrow (L_n/L)/t(L_n/L) \rightarrow 0$$

where  $T(L_n/L) := \{x \in L_n/L \mid t \cdot x = 0\}$  is an  $R[G]$ -module. Since  $L \subset tL_n$ , the map  $L_n/tL_n \rightarrow (L_n/L)/t(L_n/L)$  is an isomorphism. Hence,

$$T(L_n/L) \simeq L/tL.$$

Notice that we have an exact sequence of  $R[G]$ -modules

$$0 \rightarrow T(L_n/L) \rightarrow L_n/L \xrightarrow{t} t(L_n/L) \rightarrow 0.$$

Since  $L \subset tL_n$ ,  $t(L_n/L) = tL_n/L$ . The module  $tL_n/L$  is the kernel of the surjective map  $L_n/L \rightarrow L_n/tL_n$  between  $R[G]$ -free modules and so it is  $R[G]$ -free. Hence,  $T(L_n/L)$  is also  $R[G]$ -free. Therefore,  $L/tL$  is also  $R[G]$ -free.  $\square$

**Corollary 3.5.** *If  $L_1 \subset L_2$  are two  $A[G]$ -lattices, then  $L_1/L_2$  is a finitely generated projective  $R[G]$ -module.*

*Proof.* There is  $n \geq 0$  such that  $t^n L_0 \subset L_1 \subset L_2 \subset t^{-n} L_0$ . This gives an exact sequence

$$0 \rightarrow L_2/L_1 \rightarrow t^{-n} L_0/L_1 \rightarrow t^{-n} L_0/L_2 \rightarrow 0$$

with middle and right terms  $R[G]$ -projective. It follows that  $L_2/L_1$  is  $R[G]$ -projective.  $\square$

3.b.1. Notice that if  $\gamma$  is an  $A_t[G]$ -isomorphism of the  $A_t[G]$ -module  $\mathcal{M} = M_0 \otimes_R A_t$ , then the image  $\gamma(L_0) \subset \mathcal{M}$  is an  $A[G]$ -lattice. In particular, if  $M_0 = R[G]^n$  and  $\gamma$  is given by right multiplication by the element  $g \in \text{GL}_n(A_t[G])$ , i.e by  $\gamma(m) := m \cdot g$ , then  $L_0 \cdot g \simeq A[G]^n$  is an  $A[G]$ -lattice.

3.c. **Determinants.** We continue to assume that  $R$  is a Noetherian commutative ring. Recall the definition of the virtual category  $V(R[G])$  of finitely generated projective (left)  $R[G]$ -modules from [13] (see also [5]). This is a commutative Picard category (i.e a symmetric monoidal category in which all arrows are invertible and all objects have inverses). Any finitely generated projective  $R[G]$ -module  $P$  gives an object in  $V(R[G])$ , which we will denote by  $[P]$ . The inverse of  $[P]$  is denoted by  $-[P]$ . As in [13] we will denote the monoidal structure additively. The set of isomorphism classes of objects in  $V(R[G])$  is a group which is identified with  $K_0(R[G])$ ; the group of automorphisms of the zero object  $[0]$  is identified with  $K_1(R[G])$ . If  $R = K$  is a field and  $G = \{1\}$ ,  $V(R[G]) = V(K)$  can be identified with the Picard category  $\text{Pic}_K^{\mathbb{Z}}$  of “ $\mathbb{Z}$ -graded  $K$ -lines”. Recall that the objects of  $\text{Pic}_K^{\mathbb{Z}}$  are pairs  $(L, n)$  of a  $K$ -line  $L$  and an integer  $n$  and the monoidal structure is given by

$$(L, n) + (M, m) = (L \otimes_K M, n + m).$$

The identification above is then given by sending  $P$  to  $(\det(P), \text{rank}(P))$ .

Consider the (full) subcategory  $D^b(R[G])$  of the derived category  $D(R[G])$  of the homotopy category of complexes of  $R[G]$ -modules which are bounded below, whose objects are perfect complexes. Recall that there is a “determinant” functor

$$\det : D^b(R[G]) \rightarrow V(R[G])$$

which takes the value  $[P]$  on complexes  $P[0] : \cdots \rightarrow 0 \rightarrow P \rightarrow 0 \rightarrow \cdots$  consisting of a finitely generated projective  $R[G]$ -module placed in degree 0. The functor  $\det$  satisfies an additivity property for “true” exact triangles, and other properties which are listed in [5]. To simplify our notations, we will sometimes write  $[P^\bullet]$  instead of  $\det(P^\bullet)$  for the virtual object in  $V(R[G])$  associated to the perfect complex  $P^\bullet$ .

3.c.1. By definition (cf. [14], §5), a “determinant theory” on  $\mathcal{M}$  is a rule that associates to any  $A[G]$ -lattice  $L$  as above, an object  $\delta(L)$  of  $V(R[G])$  and to each pair  $L_1 \subset L_2$  of  $A[G]$ -lattices an arrow in  $V(R[G])$

$$(3.2) \quad \delta_{L_1, L_2} : \delta(L_1) + [L_2/L_1] \rightarrow \delta(L_2)$$

(with  $[L_2/L_1]$  well-defined by Corollary 3.5), such that:

If  $L_1 \subset L_2 \subset L_3$ , the obvious diagram

$$(3.3) \quad \begin{array}{ccc} \delta(L_1) + [L_2/L_1] + [L_3/L_2] & \rightarrow & \delta(L_2) + [L_3/L_2] \\ \downarrow & & \downarrow \\ \delta(L_2) + [L_3/L_2] & \rightarrow & \delta(L_3) \end{array}$$

obtained using  $\delta_{L_1, L_2}$ ,  $\delta_{L_2, L_3}$ ,  $\delta_{L_1, L_3}$  commutes, and the diagonal morphism is obtained by combining  $\delta_{L_1, L_3}$  with the arrow  $[L_2/L_1] + [L_3/L_2] \rightarrow [L_3/L_1]$  given by the exact sequence  $0 \rightarrow L_2/L_1 \rightarrow L_3/L_1 \rightarrow L_3/L_2 \rightarrow 0$ .

(In fact, we will often also find that our construction satisfies additional compatibilities for suitable base changes  $R \rightarrow R'$  as in [14], §5.)

We can see that the set of determinant theories is a torsor over the commutative Picard category  $V(R[G])$ ; in particular, if  $\delta, \delta'$  are two determinant theories, then there is an object  $Q$  of  $V(R[G])$  and arrows

$$(3.4) \quad \delta'(L) \rightarrow \delta(L) + Q$$

for each lattice  $L$  which are functorial (for inclusion of lattices).

Consider the group  $\text{Aut}(\mathcal{M})$  of  $A_t[G]$ -linear isomorphisms of  $\mathcal{M}$ . If  $L$  is an  $A[G]$ -lattice, so is its image  $\gamma L$  under  $\gamma$ . Notice that, for each pair of lattices  $L_1 \subset L_2$ , an element  $\gamma \in \text{Aut}(\mathcal{M})$  induces an arrow

$$[L_2/L_1] \rightarrow [\gamma L_2/\gamma L_1]$$

given by an actual  $R[G]$ -module isomorphism. Hence, we can see that we can “twist  $\delta$  by  $\gamma$ ” to form a new determinant theory given by  $L \mapsto \delta(\gamma L)$ . By the above, the object

$$\mathcal{V}_\gamma = \mathcal{V}_\gamma(L) = \delta(\gamma L) - \delta(L)$$

does not depend on  $L$ . This is meant in the sense that for any two lattices  $L \subset L'$  there is a well-defined arrow

$$(3.5) \quad \mathcal{V}_\gamma(L) \rightarrow \mathcal{V}_\gamma(L')$$

which respects compositions for chains of inclusions.

3.c.2. Now take  $A = R[t]$ . Suppose that  $L$  is an  $A[G]$ -lattice in  $\mathcal{M} = M_0 \otimes_R R[t, t^{-1}]$ . To that, we can associate a coherent locally projective  $\mathcal{O}_{\mathbb{P}_R^1}[G]$ -module  $\mathcal{E}(L)$  on  $\mathbb{P}_R^1$  obtained by gluing the sheaves on  $\mathbb{A}_\infty^1 = \text{Spec}(R[t^{-1}])$  and  $\mathbb{A}_0^1 = \text{Spec}(R[t])$  that correspond to  $M_0 \otimes_{\mathbb{Z}} R[t^{-1}]$  and  $L$  respectively, along the identification

$$L \otimes_{R[t]} R[t, t^{-1}] = \mathcal{M} = (M_0 \otimes_{\mathbb{Z}} R[t^{-1}]) \otimes_{R[t^{-1}]} R[t, t^{-1}].$$

Now suppose that  $\gamma$  is an  $A_t[G]$ -isomorphism of  $M_0 \otimes_R A_t$ ; this gives the  $A[G]$ -lattice  $L = \gamma(L_0)$ ,  $L_0 = M_0 \otimes_R R[t]$ . By Theorem 9.1, when  $R$  is a Dedekind ring with finite

residue fields, all coherent locally free  $\mathcal{O}_{\mathbb{P}_R^1}[G]$ -modules can be obtained as  $\mathcal{E}(L) = \mathcal{E}(\gamma(L_0))$  for a suitable  $M_0$  and  $\gamma$  as above.

3.c.3. Denote by  $R\Gamma(\mathbb{P}_R^1, \mathcal{E}(L))$  the complex in the derived category  $D(R[G])$  that calculates the cohomology of  $\mathcal{E}(L)$  over  $\mathbb{P}_R^1$ . This is quasi-isomorphic to the Čech complex

$$(3.6) \quad C^\bullet(L) : (M_0 \otimes_{\mathbb{Z}} R[t^{-1}]) \oplus L \rightarrow M_0 \otimes_{\mathbb{Z}} R[t, t^{-1}]$$

The standard argument shows that  $R\Gamma(\mathbb{P}_R^1, \mathcal{E}(L))$  is “perfect”, i.e. is in  $D^p(R[G])$ . Hence, we can set

$$\delta(L) := \det(R\Gamma(\mathbb{P}_R^1, \mathcal{E}(L))) \in V(R[G]).$$

This gives a determinant theory as above. Indeed, an inclusion  $i : L_1 \hookrightarrow L_2$  of  $A[G]$ -lattices, gives a corresponding homomorphism of sheaves  $i : \mathcal{E}(L_1) \rightarrow \mathcal{E}(L_2)$  and of Čech complexes  $i : C^\bullet(L_1) \rightarrow C^\bullet(L_2)$ . Notice that there is a short exact sequence of complexes

$$0 \rightarrow C^\bullet(L_1) \xrightarrow{i} C^\bullet(L_2) \rightarrow (L_2/L_1)[0] \rightarrow 0$$

of  $R[G]$ -modules. Using this, we obtain a true triangle in  $D^p(R[G])$

$$R\Gamma(\mathbb{P}_R^1, \mathcal{E}(L_1)) \rightarrow R\Gamma(\mathbb{P}_R^1, \mathcal{E}(L_2)) \rightarrow (L_2/L_1)[0] \rightarrow R\Gamma(\mathbb{P}_R^1, \mathcal{E}(L_1))[1].$$

This induces the isomorphism

$$\delta_{L_1, L_2} : \delta(L_1) + [L_2/L_1] \rightarrow \delta(L_2).$$

as required. We can now see that the required properties of  $\delta$  follow from the corresponding properties of  $\det$ .

3.d. **A central extension.** Consider the group  $\text{Aut}(\mathcal{M})$  of  $R[G][t, t^{-1}]$ -linear isomorphisms of  $\mathcal{M} = M_0 \otimes_R R[t, t^{-1}]$ . Following ideas in [14] or [3] we construct the “canonical”  $V(R[G])$ -extension  $\text{Aut}(\mathcal{M})^\vee$  of  $\text{Aut}(\mathcal{M})$  (in the sense of [3, A2]) associated to the determinant theory  $\delta$ . Explicitly,  $\text{Aut}(\mathcal{M})^\vee := \text{Aut}_\delta(\mathcal{M})$  is given as follows:

(i) To every  $\gamma : \mathcal{M} \rightarrow \mathcal{M}$  in  $\text{Aut}(\mathcal{M})$  we associate the object

$$\mathcal{V}_\gamma = \delta(\gamma L_0) - \delta(L_0)$$

of  $V(R[G])$ ;

(ii) To every pair of elements  $\gamma, \gamma'$  in  $\text{Aut}(\mathcal{M})$ , we associate a “composition” isomorphism

$$c_{\gamma, \gamma'} : \mathcal{V}_\gamma + \mathcal{V}_{\gamma'} \rightarrow \mathcal{V}_{\gamma \gamma'}$$

which is given as follows:

By (3.5) applied to  $L_0$  and  $\gamma' L_0$ , we have an arrow

$$\mathcal{V}_\gamma + \mathcal{V}_{\gamma'} \rightarrow (\delta(\gamma \gamma' L_0) - \delta(\gamma' L_0)) + (\delta(\gamma' L_0) - \delta(L_0)).$$

This composed with the contraction

$$(\delta(\gamma \gamma' L_0) - \delta(\gamma' L_0)) + (\delta(\gamma' L_0) - \delta(L_0)) \rightarrow \delta(\gamma \gamma' L_0) - \delta(L_0) = \mathcal{V}_{\gamma \gamma'}$$

defines  $c_{\gamma, \gamma'}$ .



We can see that the arrows  $c_{\gamma, \gamma'}$  satisfy associativity, i.e that the obvious diagrams

$$\begin{array}{ccc} (\mathcal{V}_\gamma + \mathcal{V}_{\gamma'}) + \mathcal{V}_{\gamma''} & \longrightarrow & \mathcal{V}_{\gamma\gamma'} + \mathcal{V}_{\gamma''} \\ \downarrow & & \downarrow \\ \mathcal{V}_\gamma + (\mathcal{V}_{\gamma'} + \mathcal{V}_{\gamma''}) & \rightarrow \mathcal{V}_\gamma + \mathcal{V}_{\gamma'\gamma''} \rightarrow & \mathcal{V}_{\gamma\gamma'\gamma''} \end{array}$$

formed using the  $c$ 's and the associativity constraint in  $V(R[G])$  are commutative.

Finally, we can see, using (3.4), that the  $V(R[G])$ -extension  $\text{Aut}(\mathcal{M})^\vee := \text{Aut}_\delta(\mathcal{M})$  is independent up to isomorphism (in the sense of [3, A3]) of the choice of determinant theory  $\delta$ .

3.d.1. Notice that if  $\gamma$  belongs to the subgroup  $\text{Aut}(L_0) = \text{Aut}(M_0 \otimes_R R[t]) \subset \text{Aut}(\mathcal{M})$ , or to the subgroup  $\text{Aut}(M_0 \otimes_R R[t^{-1}]) \subset \text{Aut}(\mathcal{M})$ , we have  $\mathcal{E}(\gamma L_0) = \mathcal{E}(L_0)$  as  $\mathcal{O}_{\mathbb{P}_R^1}[G]$ -sheaves on  $\mathbb{P}_R^1$  and hence  $\delta(\gamma L_0) = \delta(L_0)$ ; this gives a canonical arrow  $[0] \rightarrow \mathcal{V}_\gamma$  and the central extension  $\text{Aut}(\mathcal{M})^\flat$  splits over  $\text{Aut}(L_0)$  and also over  $\text{Aut}(M_0 \otimes_R R[t^{-1}])$ .

3.d.2. Taking isomorphism classes  $\gamma \mapsto [\mathcal{V}_\gamma]$  gives a group homomorphism

$$(3.7) \quad \chi : \text{Aut}(\mathcal{M}) \rightarrow K_0(R[G]).$$

Denote by  $\text{Aut}'(\mathcal{M})$  the kernel of  $\chi$ . Now for each  $\gamma : \mathcal{M} \rightarrow \mathcal{M}$  in  $\text{Aut}'(\mathcal{M})$ , choose an arrow  $\phi_\gamma : \mathcal{V}_1 = [0] \xrightarrow{\sim} \mathcal{V}_\gamma$  in  $V(R[G])$ . If  $\gamma, \gamma'$  are in  $\text{Aut}'(\mathcal{M})$ , using the trivializations  $\phi_\gamma, \phi_{\gamma'}, \phi_{\gamma\gamma'}$  allows us to identify the compositions  $c_{\gamma, \gamma'}$  with elements of  $K_1(R[G])$ . We can check that the associativity amounts to the fact that

$$c : \text{Aut}'(\mathcal{M}) \times \text{Aut}'(\mathcal{M}) \rightarrow K_1(R[G]); \quad (\gamma, \gamma') \mapsto c_{\gamma, \gamma'},$$

is a 2-cocycle. There is a corresponding central extension

$$(3.8) \quad 1 \rightarrow K_1(R[G]) \rightarrow \mathcal{H}_\delta(\mathcal{M}) \rightarrow \text{Aut}'(\mathcal{M}) \rightarrow 1$$

which can be described more explicitly as follows:

$$(3.9) \quad \mathcal{H}_\delta(\mathcal{M}) = \{(\gamma, \phi_\gamma) \mid \gamma \in \text{Aut}'(\mathcal{M}), \phi_\gamma : \mathcal{V}_1 = [0] \rightarrow \mathcal{V}_\gamma\}$$

with multiplication defined using the cocycle  $c$  above. Again, up to isomorphism, the central extension  $\mathcal{H}_\delta(\mathcal{M})$  is independent of the choice of determinant theory. By §3.d.1, we see that the central extension  $\mathcal{H}_\delta(\mathcal{M})$  splits over  $\text{Aut}(L_0)$  and also over  $\text{Aut}(M_0 \otimes_R R[t^{-1}])$ . (They are obviously both subgroups of  $\text{Aut}'(\mathcal{M})$ .)

3.d.3. Now take  $A = R[t]$ , so that  $A_t = R[t, t^{-1}]$  and take  $M_0 = R[G]^n$ ,  $\mathcal{M} = A_t[G]$ . Using the isomorphism

$$\text{GL}_n(A_t[G]) \rightarrow \text{Aut}(\mathcal{M}); \quad g \mapsto (m \mapsto m \cdot g^{-1})$$

we pull-back  $\text{Aut}(\mathcal{M})^\vee$  to a categorical  $V(R[G])$ -extension  $\text{GL}_n(A_t[G])^\vee$  of  $\text{GL}_n(A_t[G])$ . This in turn extends to a categorical extension  $\text{GL}(A_t[G])^\vee$  of the infinite linear group  $\text{GL}(A_t[G]) = \varinjlim_n \text{GL}_n(A_t[G])$ . Notice that the commutator subgroup  $E(A_t[G]) \subset \text{GL}(A_t[G])$  is contained in  $\varinjlim_n \text{GL}'_n(A_t[G])$ ; this allows us to assemble the extensions obtained as above from (3.8) for  $n \gg 0$  and give a central extension

$$(3.10) \quad 1 \rightarrow K_1(R[G]) \rightarrow \mathcal{H}(A_t[G]) \rightarrow E(A_t[G]) \rightarrow 1.$$

Since  $E(A_t[G])$  is a perfect group and the Steinberg extension  $\text{St}(A_t[G])$  is its universal central extension (see [46, Chapter 4.2] or [32, Section 5]), there is a (unique) group homomorphism

$$(3.11) \quad \partial : K_2(A_t[G]) \rightarrow K_1(R[G])$$

that fits in a (unique) commutative diagram

$$(3.12) \quad \begin{array}{ccccccc} 1 & \rightarrow & K_2(A_t[G]) & \rightarrow & \text{St}(A_t[G]) & \rightarrow & E(A_t[G]) \rightarrow 1 \\ & & \partial \downarrow & & \partial \downarrow & & \downarrow \\ 1 & \rightarrow & K_1(R[G]) & \rightarrow & \mathcal{H}(A_t[G]) & \rightarrow & E(A_t[G]) \rightarrow 1 \end{array}$$

with the right vertical map the identity. Observe here that by §3.d.1, the extension (3.10) splits over  $E(A[G])$ . Hence, the homomorphism  $\partial$  is trivial on the image of  $K_2(A[G])$  in  $K_2(A_t[G])$ , i.e the composition

$$(3.13) \quad K_2(A[G]) \rightarrow K_2(A_t[G]) \xrightarrow{\partial} K_1(R[G])$$

is trivial.

**Remark 3.6.** Notice that there is a 1-1 correspondence between  $R[[t]][G]$ -lattices in  $R((t))[G]^n$  and  $R[t][G]$ -lattices in  $R[t, t^{-1}][G]^n$ ; indeed, by Proposition 3.4, both these sets are in 1-1 correspondence with the union over  $n \geq 0$  of all  $R[t][G]$ -submodules of  $t^{-n}R[t][G]/t^n R[t][G] \simeq R[t][G]/(t^{2n})$  which are  $R[G]$ -projective. Hence, our determinant theory for  $R[t, t^{-1}][G]^n$  also gives a determinant theory for  $R((t))[G]^n$ . Then the above results also apply to  $A = R[[t]]$ . The corresponding central extensions (3.8) are compatible in the sense that the central extension for  $R((t))[G]^n$  pulls back to the one for  $R[t, t^{-1}][G]^n$  under  $\text{GL}'_n(R[t, t^{-1}][G]) \hookrightarrow \text{GL}'_n(R((t))[G])$ . In particular, the same argument gives a boundary  $\partial : K_2(R((t))[G]) \rightarrow K_1(R[G])$  that satisfies (3.13) as above.

**Remark 3.7.** The homomorphism  $\partial$  is a refined version of the inverse of the tame symbol. (See below.) In a previous version of this paper, a homomorphism  $K_2(A_t[G]) \rightarrow K_1(R[G])$  was constructed as a boundary map on a suitable localization sequence for K-groups using work of Neeman-Ranicki [37], [36]. This should agree with the construction given above but working out the details of this comparison is a complicated affair.

3.d.4. In this paragraph, we will consider  $R[[t]][G]$ -lattices but the construction works with  $R[t][G]$ -lattices too. Let us fix a determinant theory  $\delta$ . Suppose that  $L_1, L_2$  are two lattices and find  $N \gg 0$  such that  $t^N L_0 \subset L_1, L_2$ . Then we can see that a choice of an isomorphism  $a : \delta(L_1) \xrightarrow{\sim} \delta(L_2)$  amounts to an isomorphism

$$[a] : [L_1/t^N L_0] \xrightarrow{\sim} [L_2/t^N L_0] .$$

Indeed,  $a$  is the unique isomorphism for which the diagram

$$(3.14) \quad \begin{array}{ccc} \delta(t^N L_0) + [L_1/t^N L_0] & \xrightarrow{\delta_{L_1, t^N L_0}} & \delta(L_1) \\ \text{id} + [a] \downarrow & & \downarrow a \\ \delta(t^N L_0) + [L_2/t^N L_0] & \xrightarrow{\delta_{L_2, t^N L_0}} & \delta(L_2) \end{array}$$

commutes. The central extension

$$(3.15) \quad 1 \rightarrow K_1(R[G]) \rightarrow \mathcal{H}_\delta(R((t))[G]^n) \rightarrow GL'_n(R((t))[G]) \rightarrow 1$$

can now also be described as follows. Recall

$$\mathcal{H}_\delta(R((t))[G]^n) = \{(g, \phi_g) \mid g \in GL'_n(R((t))[G]), \phi_g : \delta(L_0) \xrightarrow{\sim} \delta(L_0 \cdot g^{-1})\}.$$

We define an operation on  $\mathcal{H}_\delta(R((t))[G]^n)$  by

$$(g, \phi_g) \star (h, \phi_h) = (g \cdot h, \phi_h^g \circ \phi_g)$$

where  $\phi_h^g$  can be defined as follows: For  $N \gg 0$ ,  $[\phi_h^g]$  is given by the composition

$$[L_0 \cdot g^{-1}/t^N L_0] \xrightarrow{\cdot g} [L_0/t^N L_0 \cdot g] \xrightarrow{[\phi_h]} [L_0 \cdot h^{-1}/t^N L_0 \cdot g] \xrightarrow{\cdot g^{-1}} [L_0 \cdot h^{-1}g^{-1}/t^N L_0]$$

where  $[\phi_h]$  is induced by  $\phi_h$  as above and  $r(g) = \cdot g$ ,  $r(g^{-1}) = \cdot g^{-1}$  are given by right multiplication. We can see that the corresponding  $\phi_h^g : \delta(L_0 \cdot g^{-1}) \rightarrow \delta(L_0 \cdot (gh)^{-1})$  is independent of the choice of  $N$ ; we will abuse notation and write

$$\phi_h^g = r(g^{-1}) \circ \phi_h \circ r(g).$$

We can now see that  $\star$  defines a group structure on  $\mathcal{H}_\delta(R((t))[G]^n)$ . The inverse of  $(g, \phi_g)$  is given by  $(g^{-1}, \psi_g)$  with

$$\psi_g = \phi_g^{-g^{-1}} = r(g) \circ \phi_g^{-1} \circ r(g^{-1})$$

(with the same abuse of notation as before).

3.d.5. Here we explain how the homomorphism  $\partial : K_2(A_t[G]) \rightarrow K_1(R[G])$  of the previous paragraph can often be calculated using the classical tame symbol.

Recall that for a field  $E$  we know by Matsumoto's theorem that  $K_2(E)$  is generated by symbols  $\{a, b\} = a \cup b$  for  $a, b \in E^\times$ . Suppose now that  $E$  supports a valuation  $v$ , with valuation ring  $\mathcal{O}$ , maximal ideal  $\mathfrak{m}$  and with residue field  $k = \mathcal{O}/\mathfrak{m}$ . Then the tame symbol

$$\tau : K_2(E) \rightarrow K_1(k) = k^\times$$

is defined by the rule that

$$(3.16) \quad \tau(\{a, b\}) = (-1)^{v(a)v(b)} \frac{a^{v(b)}}{b^{v(a)}} \bmod \mathfrak{m}.$$

**Proposition 3.8.** *Assume that  $R = F$  is a field and  $A = F[[t]]$ , so that  $A_t = F((t))$  is the field of Laurent power series, and that we take  $G = \{1\}$ . Then*

$$\partial : K_2(F((t))) \rightarrow K_1(F) = F^*$$

*constructed above is equal to the inverse of the tame symbol, i.e  $\partial = \tau^{-1}$ .*

*Proof.* See [29] for a very similar statement. We set  $E = F((t))$ ,  $\mathcal{O} = F[[t]]$  and suppose  $m \geq 3$ . Using the universality of the Steinberg central extension of the perfect group  $SL_m(E)$  we see that the conclusion is equivalent to the following statement: Our central extension

$$(3.17) \quad 1 \rightarrow F^* \rightarrow \mathcal{H}(E^m) \rightarrow SL_m(E) \rightarrow 1$$

above is isomorphic to the central extension  $\tilde{\mathcal{H}}(E^m)$  obtained by pushing out the Steinberg extension by  $\tau^{-1} : K_2(E) \rightarrow F^*$ . It follows from [19, Theorem (12.24)] and [33, Lemma 8.4] (this last reference shows that a “Steinberg 2-cocycle” is determined by its restriction to the subgroup  $\text{diag}(a, 1, \dots, 1, a^{-1})$  of the diagonal torus of  $\text{SL}_m(E)$  that corresponds to the long root) that the extension  $\tilde{\mathcal{H}}(E^m)$  is isomorphic to the canonical central  $F^*$ -extension of the loop group  $\text{SL}_m(F((t)))$  which appears in the theory of affine Kac-Moody algebras. It is well-known (cf. [1], [15]) that this “Kac-Moody extension” is the central extension  $\tilde{\mathcal{H}}(E^m)$  given by pairs  $(g, \alpha)$  with  $g \in \text{SL}_m(E)$  and  $\alpha$  a generator of the (graded) line bundle  $\langle g \cdot \mathcal{O}^m \mid \mathcal{O}^m \rangle$ , where for a  $\mathcal{O}$ -lattice  $L \subset E^m$ , we set

$$\langle L \mid \mathcal{O}^m \rangle := \det(L/t^N \mathcal{O}^m) \otimes \det(\mathcal{O}^m/t^N \mathcal{O}^m)^{-1}$$

for  $t^N \mathcal{O}^m \subset L$ . The group law is given by  $(g, \alpha) \cdot (h, \beta) = (gh, g(\beta) \otimes \alpha)$  (compare to the previous paragraph). Our central extension  $\mathcal{H}(E^m)$  is obtained in a similar manner using  $\langle \mathcal{O}^m \cdot g^{-1} \mid \mathcal{O}^m \rangle$ . Both extensions  $\mathcal{H}(E^m)$  and  $\tilde{\mathcal{H}}(E^m)$  split over  $\text{SL}_m(F[[t]])$ . By the argument of [29, Proof of (5.4.5)] we see that it is enough to show that the two extensions agree on the diagonal maximal torus  $T(F((t)))$  of  $\text{SL}_m(F((t)))$ . This fact now follows easily from  $\tau(a^{-1}, b^{-1}) = \tau(a, b)$  and the above.  $\square$

3.d.6. Here we suppose that  $\rho : R[G] \rightarrow M_d(F)$  is a representation over  $F$  where  $F$  is a field of characteristic zero. Base-changing by  $\rho$  gives an exact functor  $M \mapsto \rho(M) = M_d(F) \otimes_{\rho, R[G]} M$  from finitely generated projective  $R[G]$ -modules to  $M_d(F)$ -modules. Let  $e$  be an indecomposable idempotent of  $M_d(F)$ . Multiplying by  $e$  gives an equivalence of categories between  $M_d(F)$ -modules and  $F$ -vector spaces. This gives an equivalence of Picard categories  $V(M_d(F)) \xrightarrow{\sim} V(F) = \text{Pic}^{\mathbb{Z}}(F)$ . Sending  $M$  to  $e \cdot \rho(M)$  gives an exact functor to  $F$ -vector spaces which induces an additive functor

$$\rho : V(R[G]) \rightarrow V(F).$$

This induces on automorphisms of the identity object, the determinant (norm)

$$N(\rho) : K_1(R[G]) \rightarrow F^*.$$

Recall that we take  $A = R[t]$ , or  $A = R[[t]]$ . For simplicity, let us discuss  $A = R[[t]]$ . Notice that if  $L$  is an  $A[G]$ -lattice in  $A_t[G]^n$ , then  $e \cdot (M_d(F) \otimes_{\rho, R[G]} L)$  is an  $F[[t]]$ -lattice in  $F((t))^{nd}$ . Our construction for  $G = \{1\}$ , gives a determinant theory  $\delta_F$  for  $F[[t]]$ -lattices in  $F((t))^{nd}$ . We can see that for each  $A[G]$ -lattice  $L$  we have canonical isomorphisms

$$\rho(\delta(L)) \rightarrow \delta_F(e \cdot (M_d(F) \otimes_{\rho, R[G]} L))$$

in  $V(F)$ . We obtain a commutative diagram of central extensions

$$(3.18) \quad \begin{array}{ccccccc} 1 & \rightarrow & K_1(R[G]) & \rightarrow & \mathcal{H}(R((t)))[G]^n & \rightarrow & \text{GL}'_n(R((t))[G]) \rightarrow 1 \\ & & N(\rho) \downarrow & & \downarrow & & \downarrow \rho \\ 1 & \rightarrow & F^* & \rightarrow & \mathcal{H}(F((t))^{nd}) & \rightarrow & \text{GL}'_{nd}(F((t))) \rightarrow 1 \end{array}$$

where the bottom row can also be identified with the extension of [1] as in the proof of Proposition 3.8.

3.e. **Some  $p$ -adic limits.** In this section, we assume that  $R$  is a commutative ring in which  $p$  is a non-unit and set  $R_m = R/p^m R$ . We suppose that  $R_m$  is, for each  $m$ , a finite ring. Denote by  $\hat{R} = \varprojlim_m R_m$  the  $p$ -adic completion of  $R$ . Set

$$\hat{K}_i(R[G]) = \varprojlim_m K_i(R_m[G]) \quad \text{for } i = 0, 1, 2.$$

Observe that

$$(3.19) \quad \hat{K}_i(\hat{R}[G]) = \varprojlim_m K_i(\hat{R}_m[G]) = \varprojlim_m K_i(R_m[G]) = \hat{K}_i(R[G]).$$

Notice that the natural maps  $K_1(R_{m+1}[G]) \rightarrow K_1(R_m[G])$  are surjective since  $p^m R_{m+1}[G]$  is nilpotent in  $R_{m+1}[G]$ . Since  $R_m$  is finite for each  $m$  the natural map

$$K_1(\hat{R}[G]) \xrightarrow{\sim} \varprojlim_m K_1(R_m[G]) = \hat{K}_1(R[G])$$

is an isomorphism ([18]). Similarly, we have  $K_0(R_{m+1}[G]) \xrightarrow{\sim} K_0(R_m[G])$  and so

$$K_0(\hat{R}[G]) \xrightarrow{\sim} \hat{K}_0(R[G]).$$

Tensoring  $P \mapsto R_m \otimes_{R_{m+1}} P$  induces an additive functor

$$(3.20) \quad V(R_{m+1}[G]) \xrightarrow{r_m} V(R_m[G]) .$$

3.e.1. Set  $A = R[[t]]$ ,  $M_0 = R[G]^n$ . Also set

$$\hat{R}\{\{t\}\} = \varprojlim_m R_m((t))$$

for the  $p$ -adic completion of  $R((t))$ . Set  $\text{GL}'_n(\hat{R}\{\{t\}\}[G]) := \varprojlim_m \text{GL}'_n(R_m((t))G)$ , where  $\text{GL}'_n(R_m((t))G) = \ker(\text{GL}_n(R_m((t))G) \rightarrow K_0(R_m[G]))$ . We can see that this limit is a subgroup of  $\text{GL}_n(\hat{R}\{\{t\}\}G)$  that contains  $\text{GL}'_n(R((t))[G])$ . By applying the above (and Mittag-Leffler) we see that, for each  $n$ , there is a central extension

$$(3.21) \quad 1 \rightarrow K_1(\hat{R}[G]) \rightarrow \hat{\mathcal{H}}(\hat{R}\{\{t\}\}[G]^n) \rightarrow \text{GL}'_n(\hat{R}\{\{t\}\}G) \rightarrow 1$$

where  $\hat{\mathcal{H}}(\hat{R}\{\{t\}\}[G]^n) = \varprojlim_m \mathcal{H}(R_m((t))[G]^n)$ . This extension restricts to (3.15) after pulling back via the inclusion  $\text{GL}'_n(R((t))[G]) \hookrightarrow \text{GL}'_n(\hat{R}\{\{t\}\}[G])$ . The extensions (3.21) are compatible for various  $n$ .

By restricting to the commutator subgroup  $E(\hat{R}\{\{t\}\}[G]) \subset \text{GL}'(\hat{R}\{\{t\}\}[G])$  we obtain a central extension

$$(3.22) \quad 1 \rightarrow K_1(\hat{R}[G]) \rightarrow \hat{\mathcal{H}}(\hat{R}\{\{t\}\}[G])_E \rightarrow E(\hat{R}\{\{t\}\}G) \rightarrow 1$$

and the argument using the universality of the Steinberg extension now gives

$$\hat{\partial}_R : K_2(\hat{R}\{\{t\}\}[G]) \rightarrow K_1(\hat{R}[G]).$$

3.e.2. When  $R = \hat{R}$  is a  $p$ -adically complete discrete valuation ring, we can also construct the group  $\hat{\mathcal{H}}_n(R\{\{t\}\}[G])$  as follows: Given  $g = (g_m)_m \in \mathrm{GL}'_n(R\{\{t\}\}G)$  consider the complex of  $R[G]$ -modules

$$(3.23) \quad \hat{C}_R(g) : (R\langle\langle t^{-1} \rangle\rangle G)^n \oplus (R[[t]]G)^n \cdot g \rightarrow (R\{\{t\}\}G)^n.$$

Now for each  $m$ , we have the complex

$$(3.24) \quad C_m(g) : (R_m[t^{-1}]G)^n \oplus (R_m[[t]]G)^n \cdot g_m \rightarrow (R_m((t))G)^n$$

and  $C_m(g) = R_m \otimes_{R_{m+1}} C_{m+1}(g)$ . By the above, for each  $m$ ,  $C_m(g)$  is represented by a perfect complex  $P_m(g)$  of  $R_m[G]$ -modules. Using a standard argument (see [31, Lemma VI.13.13]), we can find such perfect complexes  $P_m(g)$  which support quasi-isomorphisms  $R_m \otimes_{R_{m+1}} P_{m+1}(g) \xrightarrow{\sim} P_m(g)$ ; then

$$\hat{P}(g) := \varprojlim_m P_m(g)$$

is a perfect complex of  $R[G]$ -modules that represents  $\hat{C}_R(g)$ . (In fact,  $\hat{C}_R(g)$  represents the cohomology of a locally free  $\mathcal{O}_{\mathbb{P}^1}[G]$ -module over  $\mathbb{P}_R^1$  obtained by patching as in [25].) Therefore,  $[\hat{C}(g)] = [\hat{P}(g)]$  makes sense in  $V(R[G])$ . For  $\mathrm{GL}'_n(R\{\{t\}\}) = \varprojlim_m \mathrm{GL}'_n(R_m((t))G)$  we can now see that

$$\hat{\mathcal{H}}(R\{\{t\}\}[G]^n) \simeq \{(g, \phi_g) \mid g \in \mathrm{GL}'_n(R\{\{t\}\}G), \phi_g : [R[G]] = [\hat{C}_R(1)] \xrightarrow{\sim} [\hat{C}_R(g^{-1})]\}.$$

By arguing as in §3.d.1, we can now see that the extension (3.21) splits over the subgroup  $\mathrm{GL}_n(R\langle\langle t^{-1} \rangle\rangle[G])$ .

3.e.3. Suppose here that  $R = \hat{R}$  is a  $p$ -adically complete discrete valuation ring with valuation  $v$ , fraction field  $F$  of characteristic zero and finite residue field  $k$ . We will assume that  $F[G]$  is split

$$F[G] \simeq \prod_i M_{m_i}(Z_i).$$

The ring  $R\{\{t\}\}$  is also a  $p$ -adically complete discrete valuation ring with residue field  $k((t))$ . We denote by  $F\{\{t\}\}$  the fraction field of  $R\{\{t\}\}$  which is

$$F\{\{t\}\} = F \otimes_R R\{\{t\}\} = \left\{ \sum_i a_i t^i \mid a_i \in F, \lim_{i \rightarrow -\infty} v(a_i) = +\infty, v(a_i) \gg -\infty \right\}.$$

Here we explain how we can also construct a central extension of  $E_n(F\{\{t\}\}[G])$  by

$$\overline{K}_1(F[G]) = \varprojlim_m K_1(F[G]) / \mathrm{Im}(K_1(R[G], (p^m))).$$

Notice here that under our assumptions we have

$$(3.25) \quad K_1(F[G]) \xrightarrow{\sim} \overline{K}_1(F[G]).$$

and we can identify  $K_1(F[G])$  and  $\overline{K}_1(F[G])$ .

In what follows, we denote by  $\mathrm{GL}_n^*(F\{\{t\}\}[G])$  the subgroup of  $g \in \mathrm{GL}_n(F\{\{t\}\}[G])$  with constant  $\mathrm{Det}$ , i.e  $\mathrm{Det}(g) \in \prod_i Z_i^\times$ .



**Lemma 3.9.** *a) If  $g \in \mathrm{GL}_n(F\{\{t\}\}[G])$  and  $m \gg 0$ , we can find  $g_m \in \mathrm{GL}_n(F \otimes_R R((t))[G])$ ,  $u_m \in \mathrm{GL}_n(R\{\{t\}\}[G], (p)^m)$ , such that  $g = u_m g_m$ .*

*b) If in addition  $g \in \mathrm{GL}_n^*(F\{\{t\}\}[G])$  then, for  $m \gg 0$ , we can write  $g = u_m g_m$  with  $g_m \in \mathrm{GL}_n^*(F \otimes_R R((t))[G]) \subset \mathrm{GL}_n'(F((t))[G])$ ,  $u_m \in \mathrm{GL}_n^*(R\{\{t\}\}[G], (p)^m)$ .*

*Proof.* Let  $\mathfrak{M} = \prod_i M_{m_i}(\mathcal{O}_{Z_i})$  be a maximal order in  $F[G] = \prod_i M_{m_i}(Z_i)$ ; Since we have  $\mathrm{GL}_n(\mathfrak{M}\{\{t\}\}, (p)^a) \subset \mathrm{GL}_n(R\{\{t\}\}[G], (p)^m)$  for  $a \gg m$  (see the proof of Lemma 2.9) we can reduce, by Morita equivalence, the proof of (a) to the case  $G = \{1\}$ . Now use the Bruhat decomposition

$$\mathrm{GL}_n(F\{\{t\}\}) = \bigcup_{(s_1, \dots, s_n) \in \mathbb{Z}^n} \mathrm{GL}_n(R\{\{t\}\}) \cdot \mathrm{diag}(\pi^{s_1}, \dots, \pi^{s_n}) \cdot \mathrm{GL}_n(R\{\{t\}\})$$

and the fact that  $\mathrm{GL}_n(R((t)))$  is dense in  $\mathrm{GL}_n(R\{\{t\}\})$  to deduce that  $\mathrm{GL}_n(F \otimes_R R((t)))$  is dense in  $\mathrm{GL}_n(F\{\{t\}\})$ ; part (a) then follows. Part (b) then follows by part (a) and an argument as in the proof of Lemma 2.9.  $\square$

Starting with  $g \in \mathrm{GL}_n^*(F\{\{t\}\}[G])$  write  $g = u_m g_m$  as in Lemma 3.9 (b) and consider  $\mathcal{V}_m := \mathcal{V}_{g_m}^{-1} = \delta(L_0 \cdot g_m^{-1}) - \delta(L_0)$  in  $V(F[G])$ . Consider the Picard category  $V_m(F[G])$  which has the same objects as  $V(F[G])$  and morphisms equivalence classes of arrows in  $V(F[G])$ , where  $a, a' : A \rightarrow B$  are equivalent if  $a' \cdot a^{-1} \in \mathrm{Im}(K_1(R[G], (p)^m)) \subset K_1(F[G])$ . Similarly, we can define  $V_m(R[G])$ . There are additive functors  $q_m : V_{m+1}(F[G]) \rightarrow V_m(F[G])$ . The object  $\mathcal{V}_m$  can be made, up to unique isomorphism in  $V_m(F[G])$ , to be independent of the choice of  $g_m$ ; Suppose another choice  $g'_m$  gives  $\mathcal{V}'_m$ . Now the central extension structure gives a canonical arrow

$$\mathcal{V}_m \rightarrow \mathcal{V}'_m + \mathcal{V}_u$$

for some  $u = g'_m g_m^{-1} \in \mathrm{GL}_n^*(R\{\{t\}\}[G], (p)^m)$ . We can think of  $\mathcal{V}_u$  as an object of  $V(R[G])$ ; it supports a unique trivialization in  $V_m(R[G])$  and this provides us with a fixed choice of a trivialization of  $\mathcal{V}_u$  in  $V_m(F[G])$ . Hence, we have given an arrow  $\mathcal{V}_m \rightarrow \mathcal{V}'_m$  in  $V_m(F[G])$  between any two choices  $\mathcal{V}_m, \mathcal{V}'_m$ . These arrows satisfy the obvious composition condition when we are dealing with three choices  $\mathcal{V}_m, \mathcal{V}'_m, \mathcal{V}''_m$ .

We can now consider the group of pairs

$$\mathcal{H}_n(F\{\{t\}\}[G])_m = \{(g, \phi_m) \mid g \in E_n(F\{\{t\}\}[G]), \phi_m : [0] \rightarrow \mathcal{V}_m \text{ in } V_m(F[G])\}$$

with group law as in §3.d.2. This gives central extensions

$$(3.26) \quad 1 \rightarrow K_1(F[G])/\mathrm{Im}(K_1(R[G], (p)^m)) \rightarrow \mathcal{H}_n(F\{\{t\}\}[G])_m \rightarrow E_n(F\{\{t\}\}[G]) \rightarrow 1.$$

The inverse limit of these, consisting of  $\{(g, (\phi_m)_m)\}$  such that  $q_m(\phi_{m+1}) = \phi_m$ , for all  $m$ , gives the desired central extension

$$(3.27) \quad 1 \rightarrow K_1(F[G]) = \overline{K}_1(F[G]) \rightarrow \hat{\mathcal{H}}_n(F\{\{t\}\}[G]) \rightarrow E_n(F\{\{t\}\}[G]) \rightarrow 1.$$

This provides us with

$$(3.28) \quad \hat{\partial}_F : K_2(F\{\{t\}\}[G]) \rightarrow K_1(F[G]).$$

Our constructions show that the diagram

$$(3.29) \quad \begin{array}{ccc} K_2(R\{\{t\}\}[G]) & \xrightarrow{\hat{\partial}_{R,m}} & K_1(R[G]/\text{Im}(K_1(R[G], (p^m))) \\ \downarrow & & \downarrow \\ K_2(F\{\{t\}\}[G]) & \xrightarrow{\hat{\partial}_{F,m}} & K_1(F[G]/\text{Im}(K_1(R[G], (p^m))). \end{array}$$

commutes. Here the vertical arrows are given by base change, and  $\hat{\partial}_{R,m}$  is obtained from  $\hat{\partial}_R$  of the previous paragraph. Therefore,  $\hat{\partial}_{F,m}$  vanishes on  $\text{Im}(K_2(R\{\{t\}\}[G], (p^m)))$ .

3.e.4. Suppose  $g \in \text{GL}_n(F\{\{t\}\}[G])$  and consider the complex

$$\hat{C}_F(g) : (F\{t^{-1}\}[G])^n \oplus (R[[t]] \otimes_R F[G])^n \cdot g \rightarrow (F\{\{t\}\}[G])^n$$

of  $F[G]$ -modules. If  $g \in \text{GL}_n(R\{\{t\}\}[G])$ , then  $\hat{C}_F(g) = F \otimes_R \hat{C}_R(g)$  with  $\hat{C}_R(g)$  the perfect complex of  $R[G]$ -modules considered in §3.e.1. In general, we have

**Proposition 3.10.** *For  $g \in \text{GL}_n(F\{\{t\}\}[G])$ ,  $\hat{C}_F(g)$  is a perfect complex of  $F[G]$ -modules.*

*Proof.* By Morita equivalence, it is enough to consider the case  $G = \{1\}$ . For simplicity, set  $A_+ = R[[t]] \otimes_R F$ ,  $A_- = F\{t^{-1}\}$ ,  $A_0 = R((t)) \otimes_R F$ . These are  $F$ -subalgebras of the field  $F\{\{t\}\}$ . We have,  $F\{\{t\}\} = A_- + A_+$ , the algebras  $A_+$ ,  $A_-$  are complete,  $A_0$  is dense in  $F\{\{t\}\}$ . Set  $G_\pm = \text{GL}_n(A_\pm)$  and  $G_0 = \text{GL}_n(A_0)$ . By Lemma 3.9 (a) and its proof,  $G_0$  is dense in  $\text{GL}_n(F\{\{t\}\})$ . Notice that we can write  $a \in F\{\{t\}\}$  as a sum  $a = a_- + a_+$  with  $a_\pm \in A_\pm$  and  $|a_+|, |a_-| \leq |a|$ , where  $|\cdot|$  denotes the  $p$ -adic absolute value on  $F\{\{t\}\}$ . It now follows by the ultrametric version of Cartan's Lemma (see for example [16, III 6.3] III 6.3, or [24]) that we can write  $g = g_0 \cdot g_+ \cdot g_-$ . Then  $g = h_0 \cdot g_-$  with  $h_0 = g_0 \cdot g_+ \in \text{GL}_n(A_0)$  since  $A_+ \subset A_0$ . We have  $\hat{C}_F(g) = \hat{C}_F(h_0)$ ; this reduces to showing the proposition for  $g \in \text{GL}_n(A_0)$ . Now also observe that if  $g' = g_+ \cdot g$  with  $g_+ \in \text{GL}_n(A_+)$ , then  $\hat{C}_F(g') \simeq \hat{C}_F(g)$ ; this allows us to restrict attention to the cosets  $\text{GL}_n(A_+) \backslash \text{GL}_n(A_0)$ ; we can see that these parametrize free rank  $n$   $A_+$ -submodules  $Q$  of  $A_0^n$  with  $t^N A_+^n \subset Q \subset t^{-N} A_+^n$  for some  $N$ . Since  $t^{-N} A_+ / t^N A_+ \simeq t^{-N} F[t] / t^N F[t]$ , the usual argument shows that each coset  $\text{GL}_n(A_+) \cdot g$  has a representative  $h$  in  $\text{GL}_n(F[t, t^{-1}])$ . Hence,  $\hat{C}_F(g)$  is isomorphic to the  $p$ -adic completion of

$$F[t^{-1}]^n \oplus A_+^n \cdot h \rightarrow A_0^n$$

and this is perfect: Indeed, we can compare this to the complex for  $h = 1$  which is quasi-isomorphic to  $F^n$  and hence is perfect; since the quotient of any two  $A_+$  lattices is a finite  $F$ -vector space the result follows.  $\square$

We can now see that  $\hat{C}_F(g)$  gives the cohomology of a rank  $n$ -vector bundle over  $\mathbb{P}_F^1$ ; this is the bundle obtained by glueing by the element  $h \in \text{GL}_n(F[t, t^{-1}])$  in the above proof. Now we can construct

$$(3.30) \quad 1 \rightarrow K_1(F[G]) \rightarrow \hat{\mathcal{H}}_n(F\{\{t\}\}[G]) \rightarrow \text{GL}_n^*(F\{\{t\}\}[G]) \rightarrow 1$$

by setting

$$(3.31) \quad \hat{\mathcal{H}}_n(F\{\{t\}\}[G]) := \{(g, \phi_g) \mid g \in \text{GL}_n^*(F\{\{t\}\}[G]), \phi_g : [F[G]] \xrightarrow{\sim} [\hat{C}_F(g^{-1})]\}.$$

Now we can check that the restriction of this extension over  $E(F\{\{t\}\}[G])$  is isomorphic to the extension (3.27) constructed in the previous paragraph. We can see from the above that this extension splits over the subgroup  $\mathrm{GL}_n(F\{t^{-1}\}[G])$  and over  $\mathrm{GL}_n(F \otimes_R R[[t]][G])$ .

**3.f. Kato's Residue map.** Recall that for  $q \geq 1$  and for a field  $N$ ,  $K_q(N)$  is generated by symbols  $\{a_1, \dots, a_q\}$  with  $a_i \in N^\times$ . Suppose that  $N$  supports a normalized additive discrete valuation  $v$ . Let  $K_*(N)$  denote the graded ring  $\bigoplus_{q \geq 0} K_q(N)$ . For  $n \geq 1$ , we let  $U^n K_*(N) = \bigoplus_{q \geq 0} U^n K_q(N)$  denote the graded  $K_*(N)$ -ideal generated by elements  $a \in N^\times = K_1(N)$  with  $v(a-1) \geq n$ . We then define

$$\tilde{K}_q(N) = \lim_{\leftarrow n} \frac{K_q(N)}{U^n K_q(N)}.$$

From [30, Lemma 1] we quote:

**Lemma 3.11.** *Let  $q \geq 0$ ,  $n \geq 1$ . If  $\hat{N}$  denotes the completion of  $N$  with respect to the valuation  $v$ , then the natural map  $N \rightarrow \hat{N}$  induces isomorphisms:*

$$\frac{K_q(N)}{U^n K_q(N)} \xrightarrow{\sim} \frac{K_q(\hat{N})}{U^n K_q(\hat{N})}, \quad \tilde{K}_q(N) \xrightarrow{\sim} \tilde{K}_q(\hat{N}).$$

**3.f.1.** We consider the case where  $F$  is a finite extension of the  $p$ -adic field  $\mathbb{Q}_p$  with valuation ring  $R$  and valuation ideal  $P = \pi R$ . Let  $N$  denote the field of fractions of the ring of power series  $R[[t]]$  for an indeterminate  $t$ , endowed with the discrete valuation associated to the height one prime ideal  $\pi R[[t]]$ . Then the completion  $\hat{N}$  is the field  $F\{\{t\}\}$  described previously. In [30, Sec. 1] Kato defines a map

$$(3.32) \quad \mathrm{Res} : \tilde{K}_2(F\{\{t\}\}) \rightarrow \tilde{K}_1(F) = F^\times.$$

By composing with the natural map  $K_2(F\{\{t\}\}) \rightarrow \tilde{K}_2(F\{\{t\}\})$  we obtain a map which we also call  $\mathrm{Res} : K_2(F\{\{t\}\}) \rightarrow F^\times$ .

From [30, Theorem 1] we know that  $\mathrm{Res}$  is continuous with respect to the topology given by the subgroups  $\{U^n K_2(N)\}$ . In fact  $\mathrm{Res}(U^n K_2(F\{\{t\}\})) \subset 1 + P^n$ . We also have

$$\mathrm{Res}(\{a, t\}) = a, \quad \text{for } a \in F^\times,$$

and  $\mathrm{Res}$  annihilates the image of  $K_2(F \otimes_R R[[t]])$  in  $K_2(F\{\{t\}\})$ . Observe that  $\partial_F$  on the image of  $K_2(F \otimes_R R((t)))$  in  $K_2(F((t)))$  is the negative of the tame symbol by Proposition 3.8. We now explain why  $\mathrm{Res}$  on the image of  $K_2(F \otimes_R R((t)))$  in  $K_2(F\{\{t\}\})$  agrees with the tame symbol: From the localization sequence (see [46, p. 294]), noting that an arbitrary non-zero element  $x \in K_2(F \otimes_R R((t)))$  can be written as  $x = t^n y$  with  $y \in F \otimes_R R[[t]]$ , we have an exact sequence

$$0 \rightarrow \mathrm{Im}(K_2(F \otimes_R R[[t]])) \rightarrow K_2(F \otimes_R R((t))) \rightarrow K_1(\mathcal{C})$$

where  $\mathcal{C}$  is the category of perfect  $F \otimes_R R((t))$ -complexes whose homology is killed by a power of  $t$ . In the usual way we get  $K_1(\mathcal{C}) \cong K_1(F)$  and the above sequence is split by mapping  $a \mapsto a \cup t$ . Therefore we get a decomposition  $K_2(F \otimes_R R((t))) = \mathrm{Im}(K_2(F \otimes_R R[[t]])) \oplus K_1(F)$  and we can see that  $\mathrm{Res}$  on  $K_2(F \otimes_R R((t)))$  is the tame symbol.

We conclude that  $\text{Res}$  is compatible with the tame symbol in the sense that we have a commutative diagram

$$(3.33) \quad \begin{array}{ccc} K_2(F((t))) & \xrightarrow{\tau_F = \partial_F^{-1}} & K_1(F) \\ \uparrow & & \uparrow \text{id} \\ K_2(F \otimes_R R((t))) & \longrightarrow & K_1(F) \\ \downarrow & & \downarrow \\ K_2(F\{\{t\}\}) & \xrightarrow{\text{Res}} & \widehat{K}_1(F). \end{array}$$

induced by the natural inclusions.

**3.f.2. Compatibility of  $\hat{\partial}^{-1}$  with  $\text{Res}$ .** Combining Proposition 3.8 with (3.33) above we will show:

**Proposition 3.12.** *The maps  $\hat{\partial}_F^{-1}$  and  $\text{Res}$  agree on  $K_2(F\{\{t\}\})$ .*

*Proof.* Let  $\hat{x} \in K_2(F\{\{t\}\})$ . We omit the subscript  $F$ . We shall show that for arbitrary  $n > 0$  we have

$$(3.34) \quad \bar{\partial}(\hat{x}) \equiv \text{Res}(\hat{x})^{-1} \pmod{\pi^n R}$$

so that we can conclude

$$(3.35) \quad \partial(\hat{x}) = \text{Res}(\hat{x})^{-1}.$$

By Matsumoto's theorem, it will suffice to take  $\hat{x} = \{\hat{x}_1, \hat{x}_2\} = \hat{x}_1 \cup \hat{x}_2$  with  $\hat{x}_i \in F\{\{t\}\}^\times$ . Since  $F\{\{t\}\}$  is the field of fractions of the discrete valuation ring  $R\{\{t\}\}$  with valuation ideal  $\pi R\{\{t\}\}$  we can write each element  $\hat{x}_i$  in the form

$$\hat{x}_i = \pi^{M_i} \hat{s}_i \text{ with } \hat{s}_i \in R\{\{t\}\}^\times.$$

Since  $F \otimes_R R((t))$  is dense in  $F\{\{t\}\}$ , for any  $n > 0$ , we can find  $x_i^{(n)} \in (F \otimes_R R((t)))^\times$  and  $r_i^{(n)} \in R\{\{t\}\}$  so that

$$\pi^{M_i} \hat{s}_i = \hat{x}_i = x_i^{(n)} (1 + \pi^n r_i^{(n)})^{-1}.$$

By Proposition 3.8 and the above we know that both  $\hat{\partial}_F^{-1}$  and  $\text{Res}$  restrict to the tame symbol on  $K_2(F \otimes_R R((t)))$ ; so, in order to prove (3.34), it will suffice to show for  $n > 0$  that

$$(3.36) \quad \hat{\partial}(\hat{x}) \equiv \partial(x_1^{(n)} \cup x_2^{(n)}) \pmod{\pi^n}$$

$$(3.37) \quad \text{Res}(\hat{x}) \equiv \text{Res}(x_1^{(n)} \cup x_2^{(n)}) \pmod{\pi^n};$$

indeed, then we can then conclude that the two right-hand terms in (3.36) and (3.37) are equal and this will then give (3.34).

We have the equalities in  $K_2(F\{\{t\}\})$ :

$$\begin{aligned} x_1^{(n)} \cup x_2^{(n)} &= \hat{x}_1 (1 + \pi^n r_1^{(n)}) \cup \hat{x}_2 (1 + \pi^n r_2^{(n)}) \\ &= [\hat{x}_1 \cup \hat{x}_2] \cdot [(1 + \pi^n r_1^{(n)}) \cup \hat{x}_2] \cdot [\hat{x}_1 \cup (1 + \pi^n r_2^{(n)})] \cdot [(1 + \pi^n r_1^{(n)}) \cup (1 + \pi^n r_2^{(n)})]. \end{aligned}$$

Since the three last terms above belong to  $U^n K_2(F\{\{t\}\})$ , we conclude that

$$\text{Res}(\hat{x}) \equiv \text{Res}(x_1 \cup x_2) \equiv \text{Res}(x_1^{(n)} \cup x_2^{(n)}) \pmod{\pi^n}.$$

Now since  $(1 + \pi^n r_1^{(n)}) \cup (1 + \pi^n r_2^{(n)}) \in \text{Im}(K_2(R\{\{t\}\}, \pi^n))$ , from §3.e.3 we know that

$$\hat{\partial}((1 + \pi^n r_1^{(n)}) \cup (1 + \pi^n r_2^{(n)})) \equiv 1 \pmod{\pi^n}.$$

Next we observe that

$$\begin{aligned} \hat{x}_1 \cup (1 + \pi^n r_2^{(n)}) &= \pi^{M_1} \hat{s}_1 \cup (1 + \pi^n r_2^{(n)}) \\ &= (\pi^{M_1} \cup (1 + \pi^n r_2^{(n)})) + (\hat{s}_1 \cup (1 + \pi^n r_2^{(n)})). \end{aligned}$$

As previously, since  $\hat{s}_1 \cup (1 + \pi^n r_2^{(n)}) \in \text{Im}(K_2(R\{\{t\}\}, \pi^n))$ , we know that

$$\hat{\partial}(\hat{s}_1 \cup (1 + \pi^n r_2^{(n)})) \equiv 1 \pmod{\pi^n}$$

and so, in order to prove the proposition, it will suffice to show:

**Lemma 3.13.** *For  $r \in R\{\{t\}\}$  we have  $\hat{\partial}(\pi \cup (1 + \pi r)) = 1$ .*

*Proof.* First we recall the construction of the cup product  $\pi \cup u \in K_2(F\{\{t\}\})$  for  $u \in F\{\{t\}\}^\times$ . As in Section 8 of [32], we form

$$d = \begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e = \begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u^{-1} \end{pmatrix} \in \text{SL}_3(F\{\{t\}\})$$

and we note that these two elements commute in  $\text{SL}_3(F\{\{t\}\})$ ; we then choose lifts  $\tilde{d}, \tilde{e}$  in the Steinberg group of  $F\{\{t\}\}$  and  $\pi \cup u$  is defined to be the commutator

$$\pi \cup u = [\tilde{d}, \tilde{e}] \in K_2(F\{\{t\}\}).$$

From (3.27) we have the exact sequence

$$(3.38) \quad 1 \rightarrow K_1(F) \rightarrow \hat{\mathcal{H}}(F\{\{t\}\}) \rightarrow \text{SL}(F\{\{t\}\}) \rightarrow 1$$

and we recall that elements of  $\hat{\mathcal{H}}(F\{\{t\}\})$  are coherent sequences of pairs  $(g, \phi_m)$  with  $g \in \text{SL}(F\{\{t\}\})$ , where we choose  $g_m \in \text{SL}(F((t)))$  with  $gg_m^{-1} \in 1 + \pi^m M(R\{\{t\}\})$  and  $\phi_m$  an isomorphism for  $N \gg 0$

$$\phi_m : \frac{L_0}{t^N L_0} \xrightarrow{\simeq} \frac{L_0 g_m^{-1}}{t^N L_0}.$$

Recall that the elements of  $\hat{\mathcal{H}}(F((t)))$  multiply by the rule

$$(3.39) \quad (\gamma, \phi_\gamma) \star (\delta, \phi_\delta) = (\gamma\delta, \phi_\delta^\gamma \circ \phi_\gamma)$$

where, writing  $r(\gamma)$  for right multiplication by  $\gamma$ ,  $\phi_\delta^\gamma = r(\gamma^{-1}) \circ \phi_\delta \circ r(\gamma)$ ; thus, as seen previously in §3.d.4, we have  $(\gamma, \phi_\gamma)^{-1} = (\gamma^{-1}, \phi_\gamma^{-\gamma^{-1}})$ .

We now set  $u = 1 + \pi r$  for  $r \in R \setminus \{t\}$ ; we fix a positive integer  $m \geq 1$  and choose  $r_m \in R \setminus (t)$  with  $r \equiv r_m \pmod{\pi^{-1}\pi^m R}$ ; we let  $u_m = 1 + \pi r_m$  and define  $e_m$  to be the matrix

$$e_m = \begin{pmatrix} u_m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u_m^{-1} \end{pmatrix}.$$

Then from the above we get

$$\partial(\pi \cup u_m) = (d, \phi_d) \star (e_m, \phi_{e_m}) \star (d^{-1}, \phi_d^{-d^{-1}}) \star (e_m^{-1}, \phi_{e_m}^{-e_m^{-1}})$$

in  $K_1(F)$ . However, from §3.d.1 we know that the  $\mathcal{H}$ -sequence splits on  $\mathrm{SL}(F[[t]])$ , and *a fortiori* on  $\mathrm{SL}(F)$ , and so we can take  $\phi_d = 1$ . This then gives

$$\begin{aligned} \bar{\partial}(\pi \cup (1 + \pi r)) &= \varprojlim_m (d, 1_m) \star (e_m, \phi_{e_m}) \star (d^{-1}, 1_m) \star (e_m^{-1}, \phi_{e_m}^{-e_m^{-1}}) \\ &= \varprojlim_m (de_m, \phi_{e_m}^d) \star (d^{-1}e_m^{-1}, \phi_{e_m}^{-e_m^{-1}d^{-1}}) \\ &= \varprojlim_m (de_m d^{-1}e_m^{-1}, \phi_{e_m}^{-1}\phi_{e_m}^d) \\ &= \varprojlim_m (1, \phi_{e_m}^{-1}\phi_{e_m}^d) \end{aligned}$$

and so it will suffice to show that for  $m \geq 1$

$$(3.40) \quad \phi_{e_m}^{-1}\phi_{e_m}^d \equiv 1 \pmod{p^m}, \quad \text{i.e. that} \quad \phi_{e_m}^{-1}\phi_{e_m}^d \in K_1(R, \pi^m).$$

We set  $\bar{R} = R/P$ . Let  $L_1(R) = R[[t]]^3$ ,  $L_1(F) = F[[t]]^3$ . For  $\gamma \in \mathrm{SL}_3(F[[t]])$  we put  $L_\gamma(F) = L_0(F)\gamma^{-1}$ . Then we note the following:

1) over  $F$ , we have the identifications

$$(3.41) \quad \frac{L_1(F)}{t^N L_1(F)} = \frac{L_d(F)}{t^N L_1(F)}, \quad \frac{L_{e_m d}(F)}{t^N L_1(F)} = \frac{L_{de_m}(F)}{t^N L_1(F)} = \frac{L_{e_m}(F)}{t^N L_1(F)};$$

2) because  $e_m \equiv 1 \pmod{\pi^m}$  we have a canonical isomorphism

$$\frac{L_{e_m}(\bar{R})}{t^N L_1(\bar{R})} = \frac{L_1(\bar{R})}{t^N L_1(\bar{R})}.$$

Let  $\{x_{i,j}\}$  for  $1 \leq i \leq 3$ ,  $0 \leq j < N$ , be the natural  $R$ -basis of  $L_1(R)/t^N L_1(R)$ ; let  $\{\bar{x}_{i,j}\}$  for  $1 \leq i \leq 3$ ,  $0 \leq j < N$ , be the images of the basis elements  $\{x_{i,j}\}$  in  $L_1(\bar{R})/t^N L_1(\bar{R})$ ; then  $\{y_{i,j} = x_{ij}e_m^{-1}\}$  are elements in  $L_{e_m}(R)/t^N L_1(R)$ . We consider the  $R$ -linear map  $\phi : L_1(R)/t^N L_1(R) \rightarrow L_{e_m}(R)/t^N L_1(R)$  given by  $\phi(x_{ij}) = y_{ij}$ . By construction we have the commutative diagram:

$$(3.42) \quad \begin{array}{ccc} L_1(R)/t^N L_1(R) & \xrightarrow{\phi} & L_{e_m}(R)/t^N L_1(R) \\ \downarrow & & \downarrow \\ L_1(\bar{R})/t^N L_1(\bar{R}) & = & L_1(\bar{R})/t^N L_1(\bar{R}). \end{array}$$

Since there is an  $\bar{R}$ -isomorphism between  $L_1(\bar{R})/t^N L_1(\bar{R})$  and  $L_{e_m}(\bar{R})/t^N L_1(\bar{R})$  they have the same  $\bar{R}$ -rank. Therefore, by Nakayama's lemma,  $\phi$  is an isomorphism.

To conclude we evaluate  $\phi^{-1} \circ \phi^d(x_{ij})$ . First observe that  $x_{ij}d = \pi^{\varepsilon_i}x_{ij}$ , where  $\pi^{\varepsilon_i} = \pi, \pi^{-1}, 1$  if  $i = 1, 2, 3$ . Therefore,

$$y_{ij}d^{-1} = x_{ij}e_m^{-1}d^{-1} = x_{ij}d^{-1}e_m^{-1} = \pi^{-\varepsilon_i}x_{ij}e_m^{-1} = \pi^{-\varepsilon_i}y_{ij}.$$

We obtain

$$\begin{aligned} (\phi^{-1} \circ \phi^d)(x_{ij}) &= \phi^{-1} \circ r(d^{-1}) \circ \phi \circ r(d)(x_{ij}) \\ &= \phi^{-1} \circ r(d^{-1}) \circ \phi(\pi^{\varepsilon_i}x_{ij}) \\ &= \phi^{-1} \circ r(d^{-1})(\pi^{\varepsilon_i}y_{ij}) \\ &= \phi^{-1}(\pi^{\varepsilon_i}y_{ij}d^{-1}) \\ &= x_{ij}. \end{aligned}$$

which proves (3.40) as desired.  $\square$

This now also concludes the proof of Proposition 3.12.  $\square$

#### 4. PUSHDOWN MAPS AND RECIPROCITY LAWS.

In what follows, we will assume that the group algebra  $\mathbb{Q}[G] = \prod_i M_{n_i}(Z_i)$  splits as in (0.1). The extension  $Z_i/\mathbb{Q}$  is unramified at all places that do not divide the order of the group  $G$ . By Morita equivalence we have an isomorphism

$$K_1(\mathbb{Q}[G]) \xrightarrow{\sim} \prod_i K_1(Z_i) = \prod_i Z_i^\times.$$

**4.a. The definition of pushdown.** Under the above assumptions we give one of the main constructions of this paper.

4.a.1. Assume that the projective regular arithmetic surface  $Y \rightarrow \text{Spec}(\mathbb{Z})$  satisfies hypothesis (H) of the introduction. We fix a Parshin triple  $\{\eta_0, \eta_1, \eta_2\}$  on  $Y$ . As seen previously in Section 1,  $\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}$  is a finite product of discrete valuation rings

$$(4.1) \quad \hat{\mathcal{O}}_{Y, \eta_1 \eta_2} = \begin{cases} \prod_\alpha \mathbb{Q}_p(\eta_{1\alpha})[[t_\alpha]], & \text{if } \eta_1 \text{ is horizontal} \\ \prod_\beta W(k(\eta_{2\beta}))\{\{t_\beta\}\}, & \text{if } \eta_1 \text{ is vertical.} \end{cases}$$

and  $\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}$  is a finite product of fields

$$(4.2) \quad \hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2} = \begin{cases} \prod_\alpha \mathbb{Q}_p(\eta_{1\alpha})(t_\alpha), & \text{if } \eta_1 \text{ is horizontal} \\ \prod_\beta \mathbb{Q}_p(\eta_{2\beta})\{\{t_\beta\}\}, & \text{if } \eta_1 \text{ is vertical.} \end{cases}$$

4.a.2. We define the push down

$$(4.3) \quad f_{*\eta_0 \eta_1 \eta_2} : K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}[G]) \rightarrow K_1(\mathbb{Q}_p[G])$$

as follows:



i) If  $\eta_1$  is horizontal, we consider the composite of

$$K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}[G]) = \prod_{\alpha} K_2(\mathbb{Q}_p(\eta_{1\alpha})(t_{\alpha}))[G]) \xrightarrow{\partial^{-1}} \prod_{\alpha} K_1(\mathbb{Q}_p(\eta_{1\alpha})[G])$$

with

$$\text{res} : \prod_{\alpha} K_1(\mathbb{Q}_p(\eta_{1\alpha})[G]) \rightarrow K_1(\mathbb{Q}_p[G])$$

where  $\text{res}$  is the restriction (norm) map given by viewing  $\prod_{\alpha} \mathbb{Q}_p(\eta_{1\alpha})[G]$  as a finite dimensional  $\mathbb{Q}_p[G]$ -algebra.

ii) If  $\eta_1$  is vertical, then  $f_{*\eta_0 \eta_1 \eta_2}$  is the composite of

$$K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}[G]) = \prod_{\beta} K_2(\mathbb{Q}_p(\eta_{2\beta})\{\{t_{\beta}\}\}[G]) \xrightarrow{\hat{\partial}^{-1}} \prod_{\beta} K_1(\mathbb{Q}_p(\eta_{2\beta})[G])$$

with

$$\prod_{\beta} K_1(\mathbb{Q}_p(\eta_{2\beta})[G]) \xrightarrow{\text{res}} K_1(\mathbb{Q}_p[G])$$

where  $\text{res}$  is again the restriction map as above.

Notice that we are using the *inverses*  $\partial^{-1}$ ,  $\hat{\partial}^{-1}$  which by Proposition 3.8 and Proposition 3.12 agree with the tame symbol, resp. Kato's Res map. Let us remark here that  $f_{*\eta_0 \eta_1 \eta_2}$  is independent of the choice of uniformizers  $t_{\alpha}$ ,  $t_{\beta}$ . In the case that  $\eta_1$  is horizontal, this follows from Proposition 3.8 and the corresponding independence of the tame symbol, and in the case that  $\eta_1$  is vertical from Proposition 3.12 and [30, §2].

We now consider the restriction of  $f_{*\eta_0 \eta_1 \eta_2}$  to the image of  $K_2(\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}[G])$ :

**Proposition 4.1.** *Let  $\eta_1$  denote a codimension one point on  $Y$ , and let  $\eta_2 < \eta_1$  be a closed point of  $Y$ , with residue characteristic  $p$ . Then for  $x$  in  $K_2(\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}[G])^b$ :*

- (i)  $f_{*\eta_0 \eta_1 \eta_2}(x) = 1$ , if  $\eta_1$  is horizontal,
- (ii)  $f_{*\eta_0 \eta_1 \eta_2}(x)$  is in  $K_1(\mathbb{Z}_p[G])^b$ , if  $\eta_1$  is vertical.

*Proof.* (i) In the horizontal case  $\hat{\mathcal{O}}_{Y, \eta_1 \eta_2} = \prod_{\alpha} \mathbb{Q}_p(\eta_{1\alpha})[[t_{\alpha}]]$ , where  $\alpha$  runs over the branches of the completion  $\eta_1$  in a formal neighborhood of  $\text{Spec}(\hat{\mathcal{O}}_{Y, \eta_2})$ . Then this restriction is induced by the composite

$$K_2(\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}[G]) = \prod_{\alpha} K_2(\mathbb{Q}_p(\eta_{1\alpha})[[t_{\alpha}]] [G]) \xrightarrow{\partial^{-1}} \prod_{\alpha} K_1(\mathbb{Q}_p(\eta_{1\alpha})[G]) \xrightarrow{\text{res}} K_1(\mathbb{Q}_p[G]);$$

and so it is trivial since by (3.d.1) the  $\mathcal{H}$ -sequence for each  $\mathbb{Q}_p(\eta_{1\alpha})(t_{\alpha})[G]$  splits over  $\mathbb{Q}_p(\eta_{1\alpha})[[t_{\alpha}]] [G]$ , so that  $\partial$  is trivial on each factor  $K_2(\mathbb{Q}_p(\eta_{1\alpha})[[t_{\alpha}]] [G])$ .

(ii) In the vertical case we get

$$\begin{aligned} K_2(\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}[G]) &= \prod_{\beta} K_2(W(k(\eta_{2\beta}))\{\{t_{\beta}\}\}[G]) \xrightarrow{\hat{\partial}^{-1}} \\ &\xrightarrow{\hat{\partial}} \prod_{\beta} K_1(W(k(\eta_{2\beta}))[G]) \xrightarrow{\text{res}} K_1(\mathbb{Z}_p[G])^b \subset K_1(\mathbb{Q}_p[G]). \end{aligned}$$

□

4.a.3. Let  $\eta_1$  be a horizontal codimension one point on  $Y$ . Then from §1.b.1,  $\hat{\mathcal{O}}_{Y,\eta_0\eta_1} = \mathbb{Q}(\eta_1)((t))$  is a complete discrete valued field with residue field  $\mathbb{Q}(\eta_1)$ . We can therefore form the pushdown  $f_{*\eta_0\eta_1}$

$$f_{*\eta_0\eta_1} : K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1}[G]) = K_2(\mathbb{Q}(\eta_1)((t)))[G] \xrightarrow{\partial^{-1}} K_1(\mathbb{Q}(\eta_1)[G]) \xrightarrow{\text{res}} K_1(\mathbb{Q}[G])$$

where  $\text{res}$  is obtained as above. By the functoriality of  $\mathcal{H}$ -sequences and of the map  $\partial$  we have a commutative diagram:

$$\begin{array}{ccc} K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1}[G]) & \rightarrow & \prod_{\eta_2} K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}[G]) \\ f_{*\eta_0\eta_1} \downarrow & & \downarrow \prod f_{*\eta_0\eta_1\eta_2} \\ K_1(\mathbb{Q}[G]) & \rightarrow & K_1(\mathbb{Q}_p[G]). \end{array}$$

Here the product in the upper right-hand term extends over all  $\eta_2$ ,  $\eta_2 < \eta_1$ , with residue characteristic  $p$ . We have therefore shown:

**Theorem 4.2.** *Let  $\eta_1$  be a horizontal codimension one point of  $Y$  and suppose  $x$  is an element in  $K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1}[G])$ . Then the product*

$$\prod_{\eta_2, \eta_2 < \eta_1} f_{*\eta_0\eta_1\eta_2}(x)$$

*lies in the diagonal image  $K_1(\mathbb{Q}[G])^\flat \subset \prod_p K_1(\mathbb{Q}_p[G])$ .*

4.b. **Reciprocity on a vertical fiber.** Let  $\eta_1$  be a vertical codimension one point of  $Y$ . Suppose that the closure  $\bar{\eta}_1$  lies in the special fiber of  $Y$  over a prime number  $p$ .

**Theorem 4.3.** *For  $x \in K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1}[G])$  we have  $\prod_{\eta_2 < \eta_1} f_{*\eta_0\eta_1\eta_2}(x) = 1$  in  $K_1(\mathbb{Q}_p[G])$ , where the product extends over all closed points  $\eta_2$  on  $\bar{\eta}_1$ .*

*Proof.* Since  $\mathbb{Q}[G]$  splits, we can also write

$$\mathbb{Q}_p[G] = \prod_i M_{n_i}(L_i).$$

We will use the subscript  $L$  to denote base change of  $\mathbb{Q}$ -schemes to the field  $L$ . Similarly, we will use the subscript  $L$  to denote tensor product of  $\mathbb{Q}$ -algebras with  $L$  over  $\mathbb{Q}$ ,  $A_L = A \otimes_{\mathbb{Q}} L$ . With this convention we have a decomposition

$$\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}[G] = \prod_i M_{n_i}(\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2,L_i})$$

and so by Morita equivalence we obtain decompositions

$$(4.4) \quad K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}[G]) = \prod_i K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2,L_i}), \quad K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1}[G]) = \prod_i K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1,L_i}).$$

Notice that since  $Y$  is smooth over  $|G|$ , the base change  $Y \otimes_{\mathbb{Z}} \mathcal{O}_{L_i}$  is also regular and the morphism  $Y \otimes_{\mathbb{Z}} \mathcal{O}_{L_i} \rightarrow \text{Spec}(\mathcal{O}_{L_i})$  is also smooth over  $|G|$ , for all  $i$ . The above shows that we can reduce to showing the vanishing statement to the case when  $G = \{1\}$  and the base

is the ring of integers of a finite extension  $L$  of  $\mathbb{Q}_p$ . We fix  $i$  and let  $L = L_i$ . The proof of this now proceeds in three steps.

*Step 1.* We start with

**Lemma 4.4.** *Given  $n > 0$  and  $\hat{\kappa} \in K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1})$ , we can find  $\kappa \in K_2(\mathcal{O}_{Y, \eta_0})$  such that for each  $\eta_2 < \eta_1$ ,  $f_{* \eta_0 \eta_1 \eta_2}(\hat{\kappa})$  and  $f_{* \eta_0 \eta_1 \eta_2}(\kappa)$  in  $K_1(L) = L^\times$  are in the same coset of  $1 + p^n \mathcal{O}_L$ .*

*Proof.* The proof of this result is essentially the same as that of the first part of Proposition 3.12, and so we only sketch the details. Recall that  $\hat{\mathcal{O}}_{Y, \eta_0 \eta_1}$  is a  $p$ -adically complete field with valuation  $v$  and valuation ring  $\hat{\mathcal{O}}_{Y, \eta_1}$ . Then, by Matsumoto's theorem,  $\hat{x} = \hat{x}_1 \cup \hat{x}_2$  with  $\hat{x}_j \in \hat{\mathcal{O}}_{Y, \eta_0 \eta_1}^\times$  for  $j = 1, 2$ . Given any  $n > 0$ , by the density of  $\mathcal{O}_{Y, \eta_0}$  in  $\hat{\mathcal{O}}_{Y, \eta_0 \eta_1}$ , we can find  $\hat{x}_j^{(n)} \in \mathcal{O}_{Y, \eta_0}$  with

$$\hat{x}_j^{(n)} = \hat{x}_j \cdot (1 + p^n \hat{r}_j^{(n)}) \quad \text{where } \hat{r}_j^{(n)} \in \hat{\mathcal{O}}_{Y, \eta_1}.$$

Then we have

$$\begin{aligned} \hat{x}_1^{(n)} \cup \hat{x}_2^{(n)} &= \hat{x}_1(1 + p^n \hat{r}_1^{(n)}) \cup \hat{x}_2(1 + p^n \hat{r}_2^{(n)}) \\ &= [\hat{x}_1 \cup \hat{x}_2] \cdot [(1 + p^n \hat{r}_1^{(n)}) \cup \hat{x}_2] \cdot [\hat{x}_1 \cup (1 + p^n \hat{r}_2^{(n)})] \cdot [(1 + p^n \hat{r}_1^{(n)}) \cup (1 + p^n \hat{r}_2^{(n)})] \end{aligned}$$

and, as in Proposition 3.12 we know that the composite,

$$K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1}) \rightarrow K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}) \xrightarrow{\hat{\partial}^{-1}} K_1(L)$$

when applied to the three right-most terms, yields values congruent to one modulo  $p^n$ .  $\square$

This result allows us to restrict attention to  $x \in K_2(\mathcal{O}_{Y, \eta_0}[G])$ .

*Step 2.* Recall that for each closed point  $\xi$  on  $Y_L$  we write  $L(\xi)$  for the residue class field of  $\xi$ . Recall, (cf. 3.13)), that we have the exact sequence

$$K_2(L(\xi)[[t]]) \rightarrow K_2(L(\xi)((t))) \xrightarrow{\partial_\xi^{-1}} L(\xi)^\times \rightarrow 1.$$

We consider the various closed points  $\xi$  on  $Y_L$  specializing to  $\eta_2$ . We then have the commutative diagram:

$$\begin{array}{ccccc} K_2(K(Y)) & \rightarrow & \bigoplus_\xi K_2(L(\xi)((t)))/K_2(L(\xi)[[t]]) & \xrightarrow{\partial_\xi^{-1}} & \bigoplus_\xi L(\xi)^\times \\ (4.5) \quad \downarrow & & \downarrow \prod_\xi \text{res}_\xi & & \downarrow \prod_\xi \text{Norm}_\xi \\ K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}) & \rightarrow & K_2(L(\eta_2)((t)))/K_2(L(\eta_2)[[t]]) & \rightarrow & L(\eta_2)^\times \end{array}$$

where for each  $\xi$ ,  $\text{res}_\xi$  denotes restriction along  $L(\eta_2) = L \cdot \mathbb{Q}_p(\eta_2) \hookrightarrow L(\xi)$ . Note here that we know that the image of  $K_2(K(Y))$  lies in the direct sum (and not just the product) by the discussion in [45] after Lemma 4.1.

*Step 3.* To conclude we know from [45, Lemma 4.1], that the composite (given by the top horizontal maps in the above diagram) with the norm map

$$(4.6) \quad K_2(K(Y)) \rightarrow \bigoplus_\xi L(\xi)^\times \xrightarrow{\text{Norm}} L^\times$$

is trivial. By grouping the points  $\xi$  above a given closed point  $\eta_2$  on the special fiber of  $Y$  at  $p$  and using the above diagram, we see that the composite

$$(4.7) \quad K_2(K(Y)) \rightarrow K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}) \rightarrow L(\eta_2)^\times \xrightarrow{\text{Norm}} L^\times$$

is trivial. Therefore the composite

$$(4.8) \quad K_2(\mathcal{O}_{Y, \eta_0}) \rightarrow K_2(K(Y)) \rightarrow K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}) \rightarrow L(\eta_2)^\times \xrightarrow{\text{Norm}} L^\times$$

is also trivial, and this completes the proof.  $\square$

For future reference, we record the following:

**Corollary 4.5.** *The composition  $K_2(\mathbb{Q}_p\{t^{-1}\}[G]) \rightarrow K_2(\mathbb{Q}_p\{\{t\}\}[G]) \xrightarrow{\hat{\partial}^{-1}} K_1(\mathbb{Q}_p[G])$ , where the first map is induced by the inclusion  $\mathbb{Q}_p\{t^{-1}\} \subset \mathbb{Q}_p\{\{t\}\}$ , is trivial.*

*Proof.* Recall  $\mathbb{Q}_p\{t^{-1}\} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p\langle\langle t^{-1} \rangle\rangle$  is the free Tate algebra. We apply Theorem 4.3 above to  $\mathbb{P}^1$  over  $\text{Spec}(\mathbb{Z})$  and  $\eta_1$  the generic point of the special fiber at  $p$ , which we denote  $1_p$ . Denote by  $2_p$  the closed point given by  $(p, t)$ . For

$$x \in K_2(\mathbb{Q}_p\{t^{-1}\}[G])^\flat \subset K_2(\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_0 1_p}[G])$$

we have

$$\prod_{\eta_2} f_{* \eta_0 1_p \eta_2}(x) = 1$$

where the product extends over all closed points  $\eta_2$  on  $\bar{1}_p$ . We claim that

$$(4.9) \quad f_{* \eta_0 1_p \eta_2}(x) = 1 \quad \text{for } \eta_2 \neq 2_p.$$

This will then show that  $f_{* \eta_0 1_p 2_p}(x) = \hat{\partial}^{-1}(x) = 1$ . Suppose that with our usual notation  $\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_2} = W(k(\eta_2))[[t_{\eta_2}]]$ . Then, since  $\eta_2 \neq 2_p$ , we know that the image of  $t^{-1} \in \hat{\mathcal{O}}_{\mathbb{P}^1, 01_p \eta_2}$  actually lies in  $W(k(\eta_2))[[t_{\eta_2}]]$ ; thus the map  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p\langle\langle t^{-1} \rangle\rangle \rightarrow \mathbb{Q}_p(\eta_2)\{\{t_{\eta_2}\}\} = \hat{\mathcal{O}}_{\mathbb{P}^1, 01_p \eta_2}$  factors through  $\mathbb{Q}_p(\eta_2)[[t_{\eta_2}]]$ . The result then follows since the pushdown map  $f_{* 01_p \eta_2}$  is trivial on  $K_2(\mathbb{Q}_p(\eta_2)[[t_{\eta_2}]] [G])$ .  $\square$

**4.c. Reciprocity for codimension one points through a given closed point.** In this subsection we fix a closed point  $\eta_2$  of  $Y$  with residue field  $k(\eta_2)$  which we suppose to have characteristic  $p$ .

**Theorem 4.6.** *For  $x \in K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_2}[G])$ , we have*

$$\prod_{\eta_1, \eta_1 > \eta_2} f_{* \eta_0 \eta_1 \eta_2}(x) = 1$$

in  $K_1(\mathbb{Q}_p[G])$ .

*Proof.* As in the proof of Theorem 4.3, we can use Morita equivalence to reduce to showing that for  $x \in K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_2, L})$  we have

$$\prod_{\eta_2 \in \bar{\eta}_1} f_{* \eta_0 \eta_1 \eta_2}(x) = 1 \text{ in } K_1(L) = L^\times$$

where  $L$  is a finite extension of  $\mathbb{Q}_p$ . We let  $\hat{\eta}$  denote a height one prime ideal of  $\hat{\mathcal{O}}_{Y,\eta_2} \otimes_{\mathbb{Z}_p} \mathcal{O}_L$ . Then, as in the previous construction in Section 4.a, we can form push down maps:

$$f_{*\eta_0\hat{\eta}\eta_2} : K_2(\hat{\mathcal{O}}_{Y,\eta_0\hat{\eta}\eta_2,L}) \rightarrow K_1(L).$$

For  $\kappa \in K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_2,L})$  we consider the product

$$\prod_{\hat{\eta} \text{ horizontal} \mid \eta_2 < \hat{\eta}} f_{*\eta_0\hat{\eta}\eta_2}(\kappa) \cdot \prod_{\hat{\eta} \text{ vertical} \mid \eta_2 < \hat{\eta}} f_{*\eta_0\hat{\eta}\eta_2}(\kappa).$$

From [30, Proposition 7] we know that this product converges to one in  $K_1(L)$ . In order to complete the proof it therefore remains to show that if  $\hat{\eta}$  does not arise from a codimension one point of  $Y \otimes_{\mathbb{Z}_p} \mathcal{O}_L$  (i.e. if  $\hat{\eta}$  is not globally defined), then  $f_{*\eta_0\hat{\eta}\eta_2}(\kappa) = 1$ . For simplicity, we will omit the subscript  $L$ . Recall from §1.b.3 that  $\hat{\mathcal{O}}_{Y,\eta_0\eta_2}$  is obtained by localizing the complete local ring  $\hat{\mathcal{O}}_{Y,\eta_2}$  with respect to the multiplicative set of elements  $K(Y_L)^\times$  of non-zero elements in the function field of  $Y$ . Thus for such  $\hat{\eta}$ , which do not arise as codimension one points on  $Y$ , we deduce that in fact  $\kappa \in K_2(\hat{\mathcal{O}}_{Y,\hat{\eta}\eta_2})^b$  and, as  $\hat{\eta}$  is necessarily horizontal, we know from Proposition 4.1 above that for such  $\hat{\eta}$  we have  $f_{*\eta_0\hat{\eta}\eta_2}(\kappa) = 1$ .  $\square$

**4.d. Adelic push down.** Our main aim here is to show that the pushdown maps associated to Parshin triples induce a map on the adelic restricted product group  $K'_2(\mathbb{A}_{Y,012}[G]) = \prod'_{(\eta_0,\eta_1,\eta_2)} K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}[G]) \subset \prod_{(\eta_0,\eta_1,\eta_2)} K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}[G])$  (see Definition 2.2). To be more precise, the above considerations show that we have a map on each  $K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}[G])$ . We wish to show that this extends, in a natural convergent manner, to a pushdown map

$$(4.10) \quad f_* := \prod'_{(\eta_0,\eta_1,\eta_2)} f_{*\eta_0\eta_1\eta_2} : \prod'_{(\eta_0,\eta_1,\eta_2)} K_2(\hat{\mathcal{O}}_{Y,\eta_0\eta_1\eta_2}[G]) \rightarrow \prod'_p K_1(\mathbb{Q}_p[G]).$$

For this we first need:

**Proposition 4.7.** *Let  $\eta_1$  denote a vertical codimension one point of  $Y$  and let  $\eta_2 < \eta_1$  denote a closed point which is not contained in the closure of any other vertical codimension 1 point. If  $x$  is in  $K_2(\hat{\mathcal{O}}_{Y,\eta_2}[\eta_1^{-1}][G])^b$ , then  $f_{*\eta_0\eta_1\eta_2}(x) = 1$ .*

*Proof.* From Theorem 4.6 we know that

$$\prod_{\xi_1, \eta_2 < \xi_1} f_{*\eta_0\xi_1\eta_2}(x) = 1.$$

Let  $\xi_1$  denote a codimension 1 point with  $\eta_2 < \xi_1$  which is different from  $\eta_1$ . It will suffice to show that  $f_{*\eta_0\xi_1\eta_2}(x) = 1$ . This follows from the inclusion  $\hat{\mathcal{O}}_{Y,\eta_2}[\eta_1^{-1}] \subset \hat{\mathcal{O}}_{Y,\xi_1\eta_2}$ , our assumption, and Proposition 4.1.  $\square$

**Proposition 4.8.** *For any  $x = (x_{\eta_0\eta_1\eta_2})$  in  $K'_2(\mathbb{A}_{Y,012}[G])$ , the infinite product of push-downs  $\prod f_{*\eta_0\eta_1\eta_2}(x_{\eta_0\eta_1\eta_2})$  converges to an element of the restricted product  $\prod'_p K_1(\mathbb{Q}_p[G])$ .*

*Proof.* We must show that the product  $\prod_{(\eta_1,\eta_2)} f_{*\eta_0\eta_1\eta_2}(x_{\eta_0\eta_1\eta_2})$ , where we consider closed points  $\eta_2$  of residue characteristic  $p$ , is  $p$ -adically convergent in  $K_1(\mathbb{Q}_p[G])$ , and that for

almost all  $p$ , it converges to an element of  $K_1(\mathbb{Z}_p[G])^\flat$ . We write this product as

$$(4.11) \quad \prod_{(\eta_1, \eta_2)} f_{*\eta_0\eta_1\eta_2}(x_{\eta_0\eta_1\eta_2}) = \prod_{\eta_1 \text{ horiz.}, \eta_2 < \eta_1} f_{*\eta_0\eta_1\eta_2}(x_{\eta_0\eta_1\eta_2}) \cdot \prod_{\eta_1 \text{ vert.}, \eta_2 < \eta_1} f_{*\eta_0\eta_1\eta_2}(x_{\eta_0\eta_1\eta_2}).$$

We start by considering the first product. By (PK1) together with Proposition 4.1 only a finite number of  $\eta_1$  will contribute non-zero terms; moreover each  $\bar{\eta}_1$  meets the special fiber  $Y_p$  of  $Y$  at  $p$  at a finite number of closed points, and so we see that the first product affords only a finite number of non-zero terms for each prime  $p$ . Moreover, applying (PK2) (with  $k = 1$ ) to such an  $\eta_1$ , we get that for almost all  $\eta_2 < \eta_1$

$$x_{\eta_0\eta_1\eta_2} \in K_2(\hat{\mathcal{O}}_{Y, \eta_1\eta_2}[G])^\flat \cdot K_2(\hat{\mathcal{O}}_{Y, \eta_2}[\eta_1^{-1}][G])^\flat.$$

The first term has pushdown in  $K_1(\mathbb{Z}_p[G])^\flat$ . Notice that by §1, for almost all  $\eta_2 < \eta_1$ ,  $\hat{\mathcal{O}}_{Y, \eta_2}[\eta_1^{-1}] \simeq W(k(\eta_2))[[t]][1/g_1]$  where  $g_1$  gives a local equation for  $\bar{\eta}_1$ . Using this and the construction of  $\partial$  we can now see that for almost all  $\eta_2 < \eta_1$  (in characteristic  $p$ ), the second term also has pushdown that lies in  $K_1(\mathbb{Z}_p[G])^\flat$ .

To conclude we consider the contributions to the second product for the given prime number  $p$ . So we assume that  $\eta_1 \in Y_p$  and suppose  $\eta_2 < \eta_1$ . Given a positive integer  $k$ , by (PK2) we know that the product

$$\prod_{\eta_2, \eta_2 < \eta_1} f_{*\eta_0\eta_1\eta_2}(x_{\eta_0\eta_1\eta_2})$$

may be written as a finite product multiplied by a product of terms pushed down from the groups  $K_2(\hat{\mathcal{O}}_{Y, \eta_1\eta_2}[G], (p^k))^\flat$  and  $K_2(\hat{\mathcal{O}}_{Y, \eta_2}[\eta_1^{-1}][G])^\flat$ . From §3.e.3 and the construction of the push-down, we know that

$$f_{*\eta_0\eta_1\eta_2}(K_2(\hat{\mathcal{O}}_{Y, \eta_1\eta_2}[G], (p^k))^\flat) \subset K_1(\mathbb{Z}_p[G], (p^k))^\flat.$$

By Proposition 4.7 we know that if  $\eta_2$  is a smooth point of the reduced special fiber of  $Y$  at  $p$ , then

$$f_{*\eta_0\eta_1\eta_2}(K_2(\hat{\mathcal{O}}_{Y, \eta_2}[\eta_1^{-1}][G])) = 1.$$

Since there are only a finite number of non-smooth points on the reduced special fiber, we conclude that the product is  $p$ -adically convergent. Moreover, since the special fiber is smooth for almost all  $p$ , we have also shown that, for almost all  $p$  the contribution from the second product lies in  $K_1(\mathbb{Z}_p[G])^\flat$ .  $\square$

We also have:

**Proposition 4.9.** *Consider the intersection*

$$\left( \prod_{0 \leq i < j \leq 2} K'_2(\mathbb{A}_{Y, ij}[G])^\flat \right) \cap K'_2(\mathbb{A}_{Y, 012}[G]) \subset K'_2(\mathbb{A}_{Y, 012}[G])$$

*of subgroups in the unrestricted product  $K_2(\mathbb{A}_{Y, 012}[G])$ . If  $x$  lies in this intersection, then  $f_*(x) = \prod_{(\eta_0, \eta_1, \eta_2)} f_{*\eta_0\eta_1\eta_2}(x)$  converges to an element in  $K_1(\mathbb{Q}[G])^\flat \cdot \prod_p K_1(\mathbb{Z}_p[G])^\flat$ .*

*Proof.* Suppose we write  $x = a_{01} \cdot a_{12} \cdot a_{02}$  with  $a_{ij} \in K'_2(\mathbb{A}_{Y,ij}[G])^\flat$ . By Proposition 4.8 we know that the product  $f_*(x) = \prod_{(\eta_0, \eta_1, \eta_2)} f_{*\eta_0\eta_1\eta_2}(x)$  converges in  $\prod'_p K_1(\mathbb{Q}_p[G])$ . Similarly, the product  $f_*(a_{01}) = \prod_{(\eta_0, \eta_1, \eta_2)} f_{*\eta_0\eta_1\eta_2}(a_{\eta_0\eta_1})$  converges to an element in  $K_1(\mathbb{Q}[G])^\flat$  by Theorems 4.2 and 4.3 (which assure us that for  $a_{01}$  we get contributions only for a finite number of horizontal  $\eta_1$ ). By Remark 2.3, the element  $a_{02}$  belongs to the restricted product  $K'_2(\mathbb{A}_{Y,012}[G])$  and therefore the product

$$f_*(a_{02}) = \prod_{(\eta_0, \eta_1, \eta_2)} f_{*\eta_0\eta_1\eta_2}(a_{\eta_0\eta_2}).$$

converges. In fact, using the definition of  $K'_2(\mathbb{A}_{Y,02}[G])$  and Proposition 4.7 we see that, in the above product, for a given prime  $p$ , only a finite number of pairs  $\eta_2 < \eta_1$  with  $\eta_2$  over  $p$  can give non-trivial contributions. Indeed, these are pairs of two types: either  $\eta_1$  is horizontal and on the support of the divisor  $D$  (as per definition of  $K'_2(\mathbb{A}_{Y,02}[G])$ ) and  $\eta_2$  is an intersection point of  $\bar{\eta}_1$  with the special fiber at  $p$ ; or  $\eta_1$  is vertical over  $p$  and  $\eta_2$  is a singular point of the special fiber of  $Y$ . Now by rearranging this product and using the reciprocity law for codimension one points through a closed point (Theorem 4.6) we can see that it is equal to 1. Hence we have  $f_*(a_{02}) = 1$ . We can conclude that the product

$$f_*(a_{12}) = \prod_{(\eta_0, \eta_1, \eta_2)} f_{*\eta_0\eta_1\eta_2}(a_{\eta_1\eta_2})$$

also converges to an element in  $\prod'_p K_1(\mathbb{Q}_p[G])$ . Since each individual term  $f_{*\eta_0\eta_1\eta_2}(a_{\eta_1\eta_2})$  is in  $K_1(\mathbb{Z}_p[G])^\flat$  (where  $p$  is the characteristic of  $\eta_2$ ), we conclude that  $f_*(a_{12})$  converges to an element in  $\prod_p K_1(\mathbb{Z}_p[G])^\flat$  and the result follows.  $\square$

**4.e. Push down on the equivariant second Chow group.** It follows from Proposition 4.8 and Proposition 4.9 that  $f_* = \prod_{(\eta_0, \eta_1, \eta_2)} f_{*\eta_0\eta_1\eta_2}$  induces a well-defined group homomorphism

$$\frac{K'_2(\mathbb{A}_{Y,012}[G])}{(\prod_{0 \leq i < j \leq 2} K'_2(\mathbb{A}_{Y,ij}[G])^\flat) \cap K'_2(\mathbb{A}_{Y,012}[G])} \longrightarrow \frac{\prod'_p K_1(\mathbb{Q}_p[G])}{K_1(\mathbb{Q}[G])^\flat \prod_p K_1(\mathbb{Z}_p[G])^\flat}.$$

We notice that the source of this map is naturally identified with  $\mathrm{CH}_{\mathbb{A}}^2(Y[G])$  while, by the Fröhlich description §2.c.1, the target is naturally isomorphic to  $\mathrm{Cl}(\mathbb{Z}[G])$ . Hence, we obtain a group homomorphism

$$f_* : \mathrm{CH}_{\mathbb{A}}^2(Y[G]) \rightarrow \mathrm{Cl}(\mathbb{Z}[G]).$$

## 5. TRANSITIONS MATRICES AND THE FIRST CHERN CLASS

We return to the assumptions and notations of §2.b. Suppose now that  $\mathcal{E}$  is a  $\mathcal{O}_Y[G]$ -bundle; that is to say  $\mathcal{E}$  is a locally free coherent  $\mathcal{O}_Y[G]$ -module of a given rank, which we denote by  $n$ . For each point  $\eta$  of  $Y$  we choose an  $\hat{\mathcal{O}}_{Y,\eta}[G]$ -basis  $e_\eta = \{e_\eta^h\}_{h=1}^n$  of the completed stalk  $\hat{\mathcal{E}}_\eta = \mathcal{E}_\eta \otimes_{\mathcal{O}_{Y,\eta}} \hat{\mathcal{O}}_{Y,\eta}$ . For a given Parshin triple  $\{\eta_0, \eta_1, \eta_2\}$  we then have transition maps  $\lambda_{\eta_i\eta_j} \in \mathrm{GL}_n(\hat{\mathcal{O}}_{Y,\eta_i\eta_j}[G])$  with

$$(5.1) \quad (e_{\eta_i}^h)_h = \lambda_{\eta_i\eta_j} \cdot (e_{\eta_j}^k)_k$$



for  $0 \leq i, j \leq 2$ . Note that we have the obvious relation  $\lambda_{\eta_i \eta_k} = \lambda_{\eta_i \eta_j} \cdot \lambda_{\eta_j \eta_k}$ .

### 5.a. Construction of the first Chern class.

**Theorem 5.1.** *With the above notation:*

(a)  $\prod_{\eta_1} \text{Det}(\lambda_{\eta_0 \eta_1})$  lies in the restricted product  $K'_1(\mathbb{A}_{Y,01}[G]) \subset \prod_{\eta_1} K_1(\hat{\mathcal{O}}_{Y,\eta_0 \eta_1}[G])$ ; that is to say all but a finite number of terms are in  $K'_1(\hat{\mathcal{O}}_{Y,\eta_1}[G])^\flat$ .

(b) The class of  $\prod_{\eta_1} \text{Det}(\lambda_{\eta_0 \eta_1})$  in the first adelic equivariant Chow group  $\text{CH}_{\mathbb{A}}^1(Y[G])$  is independent of the choice of bases.

Before proving the theorem we use it to make the following definition:

**Definition 5.2.** *The first adelic equivariant Chern class of  $\mathcal{E}$  in  $\text{CH}_{\mathbb{A}}^1(Y[G])$ , denoted  $c_1(\mathcal{E})$ , is the class represented by  $\prod_{\eta_1} \text{Det}(\lambda_{\eta_0 \eta_1})$ .*

*Proof.* (a) The generic basis  $\{e_{\eta_0}^i\}_{i=1}^n$  is an  $\mathcal{O}_Y(U)[G]$ -basis of  $\mathcal{E}$  for some non-empty open Zariski set in  $Y$ ; this therefore gives an  $\hat{\mathcal{O}}_{Y,\eta_1}[G]$ -basis of  $\mathcal{E}$  for all but a finite number of codimension one points  $\eta_1$ ; and so  $\lambda_{\eta_0 \eta_1} \in \text{GL}_n(\hat{\mathcal{O}}_{Y,\eta_1}[G])$  for all but a finite number of codimension one points  $\eta_1$ . This shows (a).

To prove (b), let  $\{d_{\eta_i}^h\}_{h=1}^n$  denote a further system of bases for  $\mathcal{E}$ . Then

$$\begin{aligned} (d_{\eta_0}^h)_h &= \gamma_{\eta_0} \cdot (e_{\eta_0}^k)_k \text{ for } \gamma_{\eta_0} \in \text{GL}_n(\hat{\mathcal{O}}_{Y,\eta_0}[G]) \\ (d_{\eta_1}^h)_h &= \gamma_{\eta_1} \cdot (e_{\eta_1}^k)_k \text{ for } \gamma_{\eta_1} \in \text{GL}_n(\hat{\mathcal{O}}_{Y,\eta_1}[G]) \end{aligned}$$

and so for each codimension one point  $\eta_1$  of  $Y$  we have the equality

$$(d_{\eta_0}^h)_h = \gamma_{\eta_0} \cdot (e_{\eta_0}^k)_k = \gamma_{\eta_0} \lambda_{\eta_0 \eta_1} \cdot (e_{\eta_1}^j)_j = \gamma_{\eta_0} \lambda_{\eta_0 \eta_1} \gamma_{\eta_1}^{-1} \cdot (d_{\eta_1}^h)_h;$$

therefore working with the  $d$ -bases we get the new product of determinants

$$\prod_{\eta_1} \text{Det}(\gamma_{\eta_0} \lambda_{\eta_0 \eta_1} \gamma_{\eta_1}^{-1}) = \text{Det}(\gamma_{\eta_0}) \prod_{\eta_1} \text{Det}(\lambda_{\eta_0 \eta_1}) \text{Det}(\gamma_{\eta_1}^{-1})$$

and the result follows since

$$\text{Det}(\gamma_{\eta_0}) \in K_1(\hat{\mathcal{O}}_{Y,\eta_0}[G])^\flat, \quad \text{Det}(\gamma_{\eta_1}) \in K_1(\hat{\mathcal{O}}_{Y,\eta_1}[G])^\flat.$$

□

**Remark 5.3.** Let  $M$  be a locally free finitely generated  $\mathbb{Z}[G]$ -module which defines an  $\mathcal{O}_S[G]$ -bundle  $\mathcal{E} = \tilde{M}$  on  $S = \text{Spec}(\mathbb{Z})$ . The above construction gives a Chern class  $c_1(\mathcal{E})$  in

$$(5.2) \quad \text{CH}_{\mathbb{A}}^1(S[G]) = \frac{\prod'_p K_1(\mathbb{Q}_p[G])}{K_1(\mathbb{Q}[G])^\flat \prod_p K_1(\mathbb{Z}_p[G])^\flat}.$$

In this case, we can see [17] that the map  $M \mapsto c_1(\tilde{M})$  gives an isomorphism  $\text{Cl}(\mathbb{Z}[G]) \xrightarrow{\sim} \text{CH}_{\mathbb{A}}^1(S[G])$ . In what follows, we use this isomorphism to identify  $\text{Cl}(\mathbb{Z}[G])$  with  $\text{CH}_{\mathbb{A}}^1(S[G])$ ; this is the negative of the isomorphism given by the classical Fröhlich description of the class group  $\text{Cl}(\mathbb{Z}[G])$  (cf. §2.c.1).

## 6. ELEMENTARY STRUCTURES AND THE SECOND CHERN CLASS

The definition of the second adelic Chern class is more involved. We first need some prerequisites.

## 6.a. The Steinberg extension.

Let  $R$  denote a commutative ring and  $G$  a finite group. Recall from §2.a that  $\mathrm{GL}(R[G])$  denotes the full general linear group over the group ring  $R[G]$  and  $\mathrm{E}(R[G])$  is the subgroup of elementary matrices with entries in  $R[G]$ . Recall from [46, Chapter 4.2] and [32, Section 5] that we have the Steinberg group  $\mathrm{St}(R[G])$  which sits in the central exact sequence

$$(6.1) \quad 1 \rightarrow \mathrm{K}_2(R[G]) \rightarrow \mathrm{St}(R[G]) \rightarrow \mathrm{E}(R[G]) \rightarrow 1.$$

6.a.1. Suppose  $G$  is a central group extension

$$1 \rightarrow A \rightarrow G \rightarrow H \rightarrow 1$$

of a group  $H$  by a group  $A$ . If  $c, d$  are two elements of  $H$ , we may choose lifts (preimages)  $\tilde{c}, \tilde{d}, \tilde{cd}$ , of the elements  $c, d, cd$  of  $H$  in  $G$ . We can then define

$$z := \tilde{cd} \cdot (\tilde{d})^{-1} \cdot (\tilde{c})^{-1} \in A.$$

Although  $z$  depends on the choice of lifts of these three elements, we will sometimes abuse notation and write the element  $z$  as  $z(c, d)$ . We will refer to these elements as “cocycles” since if  $s : H \rightarrow G$  is a set-theoretic section of  $G \rightarrow H$ , then the map  $z : H \times H \rightarrow A$  given by  $z(c, d) = s(cd) \cdot s(d)^{-1} \cdot s(c)^{-1}$  gives a 2-cocycle with values in  $A$ .

The following lemma will be applied repeatedly to the Steinberg central exact sequence (6.1) and its variants.

**Lemma 6.1.** *Suppose  $b, c, d \in H$ . Fix a choice of a pre-image  $\tilde{h} \in G$  of each element  $h$  in the sequence  $c, d, b, cd, db, cdb$ . These choices allow us to define as above the elements  $z(cd, b)$ ,  $z(c, d)$ ,  $z(c, db)$ ,  $z(d, b)$  of  $A$  and we have*

$$(6.2) \quad z(cd, b)z(c, d) = z(c, db)z(d, b).$$

*Proof.* This is just the two-cocycle relation of [47, Chapter VII.3, p. 121] since  $A$  is central in  $G$ . Notice here that we do not require that the lifts of two elements in the sequence that are equal are also equal.  $\square$

6.b. **Elementary structure.** We continue with the assumptions and notations of §2.b.

**Definition 6.2.** *Suppose  $\mathcal{E}$  is a  $\mathcal{O}_Y[G]$ -bundle. An (adelic) elementary structure on  $\mathcal{E}$  is an equivalence class of choices of  $\hat{\mathcal{O}}_{Y, \eta_i}[G]$ -bases  $\{e_{\eta_i}^h\}_h$  of  $\hat{\mathcal{E}}_{\eta_i} = \mathcal{E} \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{Y, \eta_i}$  with the property that for each Parshin pair  $\{\eta_i, \eta_j\}$  on  $Y$  the image of the transition map  $\lambda_{\eta_i \eta_j}$  in  $\mathrm{GL}(\hat{\mathcal{O}}_{Y, \eta_i \eta_j}[G])$  lies in the subgroup of elementary matrices  $\mathrm{E}(\hat{\mathcal{O}}_{Y, \eta_i \eta_j}[G])$ . Here, another set of  $\hat{\mathcal{O}}_{Y, \eta_i}[G]$ -bases  $\{d_{\eta_i}^j\}_j$  of  $\mathcal{E}$  is called equivalent to  $\{e_{\eta_i}^h\}_h$  if it is related to  $\{e_{\eta_i}^h\}_h$  by an elementary base change i.e., when we write  $d_{\eta_i} = \mu_{\eta_i} e_{\eta_i}$ , then  $\mu_{\eta_i}$  is elementary in the sense that  $\mu_{\eta_i} \in \mathrm{E}(\hat{\mathcal{O}}_{Y, \eta_i}[G])$ .*

**Remark 6.3.** a) In what follows, we will omit the adjective “adelic” and talk simply of elementary structures.

b) Having an elementary structure is a stable property, i.e an  $\mathcal{O}_Y[G]$ -bundle  $\mathcal{E}$  has elementary structure if and only if the bundle  $\mathcal{E} \oplus \mathcal{O}_Y[G]^n$  does for some  $n \geq 0$ .

c) If  $\mathcal{E}$  supports an elementary structure as above, then its first adelic Chern class  $c_1(\mathcal{E})$  is trivial. Indeed, we then have  $\text{Det}(\lambda_{\eta_0\eta_1}) = 1$ . The converse is not necessarily always true; roughly speaking, the reason is the non-vanishing of various  $\text{SK}_1$  terms which are not detected by  $c_1(\mathcal{E})$ . It is however, true when  $G$  is trivial, see below. (Recall that we are assuming that the arithmetic surface  $Y$  is regular.)

d) If the group  $G$  is trivial, then one can easily see that an  $\mathcal{O}_Y$ -bundle has an elementary structure, if and only if the determinant  $\det(\mathcal{E})$  is a trivial line bundle. In that case,  $\mathcal{E}$  supports a unique elementary structure.

### 6.c. The second adelic Chern class.

The second adelic Chern class will only be defined for bundles with elementary structure on suitable arithmetic surfaces. We suppose assumption (H) on  $Y$  is satisfied; we also continue to suppose that the group algebra  $\mathbb{Q}[G]$  splits.

6.c.1. We start by showing the following properties of elementary transitions. If  $\lambda_{\eta_i\eta_j}$  is an element of  $E(\hat{\mathcal{O}}_{Y,\eta_i\eta_j}[G])$  we will denote by  $\tilde{\lambda}_{\eta_i\eta_j}$  an element of  $\text{St}(\hat{\mathcal{O}}_{Y,\eta_i\eta_j}[G])$  that projects to  $\lambda_{\eta_i\eta_j}$ . For simplicity, we will sometimes omit the subscript  $Y$  from the notation.

**Proposition 6.4.** *Suppose that  $\mathcal{E}$  is an  $\mathcal{O}_Y[G]$ -bundle with elementary structure given by the bases  $\{e_{\eta_i}^h\}_h$  with corresponding transition matrices  $\lambda_{\eta_i\eta_j}$  in  $E(\hat{\mathcal{O}}_{\eta_i\eta_j}[G])$ . Then we also have:*

- 1) *There is an effective divisor  $D$  on  $Y$  containing all vertical fibers over primes that divide the order of  $G$  such that for every Parshin triple  $(\eta_0, \eta_1, \eta_2)$  on  $Y$  we have:*
  - a)  $\lambda_{\eta_0\eta_1}$  belongs to  $E(\hat{\mathcal{O}}_{\eta_1}[D^{-1}][G]) \subset E(\hat{\mathcal{O}}_{\eta_0\eta_1}[G])$ ,
  - b)  $\lambda_{\eta_0\eta_2}$  belongs to  $E(\hat{\mathcal{O}}_{\eta_2}[D^{-1}][G]) \subset E(\hat{\mathcal{O}}_{\eta_0\eta_2}[G])$ .
- 2) *For all lifts  $\tilde{\lambda}_{\eta_i\eta_j}$  of  $\lambda_{\eta_i\eta_j}$  with  $\tilde{\lambda}_{\eta_0\eta_1}$  in  $\text{St}(\hat{\mathcal{O}}_{\eta_1}[D^{-1}][G])$ ,  $\tilde{\lambda}_{\eta_1\eta_2}$  in  $\text{St}(\hat{\mathcal{O}}_{\eta_1\eta_2}[G])$ , and  $\tilde{\lambda}_{\eta_0\eta_2}$  in  $\text{St}(\hat{\mathcal{O}}_{\eta_2}[D^{-1}][G])$ , the element*

$$z := \prod_{(\eta_0, \eta_1, \eta_2)} \tilde{\lambda}_{\eta_0\eta_2} \cdot (\tilde{\lambda}_{\eta_0\eta_1})^{-1} \cdot (\tilde{\lambda}_{\eta_1\eta_2})^{-1}$$

*lies in the group*

$$K'_2(\mathbb{A}_{Y,012}[G]) \cdot K_2(\mathbb{A}_{Y,12}[G])^\flat \cdot K'_2(\mathbb{A}_{Y,01}[G])^\flat.$$

*Proof.* Recall  $Y$  is integral so there is only one  $\eta_0$  which we denote by 0. We will first show (1). Since  $\hat{\mathcal{O}}_{\eta_0}$  is the function field  $K(Y)$  of  $Y$ , there is a divisor  $D \subset Y$  such that the  $e_0^h$  give an  $\mathcal{O}_U[G]$ -basis of  $\mathcal{E}|_U$ , where  $U$  is the open complement of  $D$  in  $Y$ . For simplicity, we will omit the superscript  $h$ . By increasing  $D$ , we may assume that  $D$  contains all the vertical fibers of  $Y$  over primes that divide the order of the group  $G$ . Now notice that if  $\eta_i$ ,  $i = 1, 2$ , lie in  $U$ , then both  $e_0$  and  $e_{\eta_i}$  are bases of  $\mathcal{E}$  over  $\hat{\mathcal{O}}_{\eta_i}$  and so the transition  $\lambda_{\eta_0\eta_i}$  lies in the intersection  $E(\hat{\mathcal{O}}_{\eta_0\eta_i}[G]) \cap \text{GL}(\hat{\mathcal{O}}_{\eta_i}[G])$ . Since the order of the group  $G$  is invertible here,

we have by Morita equivalence and Corollary 2.8,  $E(\hat{\mathcal{O}}_{\eta_0\eta_i}[G]) = \mathrm{SL}(\hat{\mathcal{O}}_{\eta_0\eta_i}[G])$ ,  $E(\hat{\mathcal{O}}_{\eta_i}[G]) = \mathrm{SL}(\hat{\mathcal{O}}_{\eta_i}[G])$ . Hence,  $\lambda_{0\eta_i}$  lies in  $E(\hat{\mathcal{O}}_{\eta_i}[G]) = E(\hat{\mathcal{O}}_{\eta_i}[D^{-1}][G])$  as required. Now suppose that  $\eta_i$ ,  $i = 1, 2$  is on  $D$ . If  $i = 1$ , there is nothing to show since  $\hat{\mathcal{O}}_{\eta_1}[D^{-1}] = \hat{\mathcal{O}}_{0\eta_1}$ . Consider  $\eta_2$  on  $D$ . Both  $e_0$  and  $e_{\eta_2}$  are bases of  $\mathcal{E}$  over  $\hat{\mathcal{O}}_{\eta_2}[D^{-1}]$  and as above we have  $\lambda_{0\eta_2} \in E(\hat{\mathcal{O}}_{0\eta_2}[G]) \cap \mathrm{GL}(\hat{\mathcal{O}}_{\eta_2}[D^{-1}][G])$ . The order of  $G$  is invertible in the rings  $\hat{\mathcal{O}}_{0\eta_2}$ ,  $\hat{\mathcal{O}}_{\eta_2}[D^{-1}]$  and we are assuming that the group algebra  $\mathbb{Q}[G]$  splits. Hence, we can apply Morita equivalence and Corollary 2.8 to show that the group rings  $\hat{\mathcal{O}}_{0\eta_2}[G]$ ,  $\hat{\mathcal{O}}_{\eta_2}[D^{-1}][G]$ , have trivial  $\mathrm{SK}_1$ . Hence, as above, the intersection  $E(\hat{\mathcal{O}}_{0\eta_2}[G]) \cap \mathrm{GL}(\hat{\mathcal{O}}_{\eta_2}[D^{-1}][G])$  is equal to  $E(\hat{\mathcal{O}}_{\eta_2}[D^{-1}][G])$  and we have just shown (1).

We will now show (2). First, we will show that we can adjust the bases at codimension 1 points  $\eta_1$  to make sure that the new transition matrices  $\theta_{\eta_i\eta_j}$  have lifts  $\tilde{\theta}_{\eta_i\eta_j}$  for which (2) is satisfied. We will not change the bases for  $\eta_1$  off  $D$ . If  $\eta_1$  is a component of  $D$  we have

$$\lambda_{0\eta_1} \in E(\hat{\mathcal{O}}_{0\eta_1}[G]) = \mathrm{SL}(\hat{\mathcal{O}}_{0\eta_1}[G]).$$

By Lemma 2.9 we have

$$(6.3) \quad \mathrm{SL}(\hat{\mathcal{O}}_{0\eta_1}[G]) = \mathrm{SL}(\mathcal{O}_{0\eta_1}[G]) \cdot \mathrm{SL}(\hat{\mathcal{O}}_{\eta_1}[G]).$$

Hence, we can write  $\lambda_{0\eta_1} = \nu'_{0\eta_1} \cdot \mu'_{\eta_1}$  with  $\mu'_{\eta_1} \in \mathrm{SL}(\hat{\mathcal{O}}_{\eta_1}[G])$ ,  $\nu'_{0\eta_1} \in \mathrm{SL}(\mathcal{O}_{0\eta_1}[G])$ . Now use that by Corollary 2.11 the natural map

$$(6.4) \quad \mathrm{SK}_1(\mathcal{O}_{\eta_1}[G]) \rightarrow \mathrm{SK}_1(\hat{\mathcal{O}}_{\eta_1}[G])$$

is surjective to find a matrix  $g \in \mathrm{SL}(\mathcal{O}_{\eta_1}[G])$  such that  $[g] = [\mu'_{\eta_1}]$  in  $\mathrm{SK}_1(\hat{\mathcal{O}}_{\eta_1}[G])$ . Our new basis at  $\eta_1$  is  $f_{\eta_1} = \mu_{\eta_1} \cdot e_{\eta_1}$  where  $\mu_{\eta_1} := g^{-1} \cdot \mu'_{\eta_1}$  is in  $E(\hat{\mathcal{O}}_{\eta_1}[G])$ . Now we can write

$$\lambda_{0\eta_1} = (\nu'_{0\eta_1} \cdot g) \cdot (g^{-1} \cdot \mu'_{\eta_1}) = \theta_{0\eta_1} \cdot \mu_{\eta_1}$$

and observe that  $\theta_{0\eta_1} := \nu'_{0\eta_1} \cdot g$  is actually in  $\mathrm{SL}(\mathcal{O}_{0\eta_1}[G]) = E(\mathcal{O}_{0\eta_1}[G]) = E(K(Y)[G])$ . We leave all the other bases  $e_{\eta_i}$  unchanged, i.e we set  $f_0 = e_0$ ,  $f_{\eta_2} = e_{\eta_2}$ , and  $f_{\eta_1} = e_{\eta_1}$  if  $\eta_1$  is not on  $D$ . Hence, if  $\eta_1$  is on  $D$ , we have:

$$(6.5) \quad \theta_{0\eta_1} = \lambda_{0\eta_1} \cdot \mu_{\eta_1}^{-1}, \quad \theta_{0\eta_2} = \lambda_{0\eta_2}, \quad \theta_{\eta_1\eta_2} = \mu_{\eta_1} \cdot \lambda_{\eta_1\eta_2}.$$

On the other hand, if  $\eta_1$  is not on  $D$ , the transitions do not change:

$$(6.6) \quad \theta_{0\eta_1} = \lambda_{0\eta_1}, \quad \theta_{0\eta_2} = \lambda_{0\eta_2}, \quad \theta_{\eta_1\eta_2} = \lambda_{\eta_1\eta_2}.$$

The new bases  $f_{\eta_i}$  give an equivalent elementary structure; we can also see that they satisfy (1) for the same divisor  $D$ . We will now explain how to pick lifts  $\tilde{\theta}_{\eta_i\eta_j}$  of  $\theta_{\eta_i\eta_j}$  so that the corresponding cocycle  $z(\tilde{\theta}) = \prod_{(0,\eta_1,\eta_2)} \tilde{\theta}_{0\eta_2} \cdot (\tilde{\theta}_{\eta_1\eta_2})^{-1} \cdot (\tilde{\theta}_{0\eta_1})^{-1}$  lies in  $K'_2(\mathbb{A}_{012}[G]) \cdot K_2(\mathbb{A}_{Y,12}[G])^\flat$ .

a) Suppose  $\eta_1$  is not on  $D$ . Since  $\hat{\mathcal{O}}_{\eta_2}[D^{-1}] \subset \hat{\mathcal{O}}_{\eta_1\eta_2}$ , and  $\hat{\mathcal{O}}_{\eta_1} \subset \hat{\mathcal{O}}_{\eta_1\eta_2}$  we can view all transitions  $\theta_{0\eta_1}$ ,  $\theta_{0\eta_2}$ ,  $\theta_{\eta_1\eta_2}$  as elements of  $E(\hat{\mathcal{O}}_{\eta_1\eta_2}[G])$ ; we pick lifts  $\tilde{\theta}_{\eta_i\eta_j}$  in  $\mathrm{St}(\hat{\mathcal{O}}_{\eta_1\eta_2}[G])$  and we can see that at such triples  $(0, \eta_1, \eta_2)$  we have

$$(6.7) \quad z(\tilde{\theta})_{(0,\eta_1,\eta_2)} \in K_2(\hat{\mathcal{O}}_{\eta_1\eta_2}[G]).$$

b) If  $\eta_1$  is on  $D$ , then by the above the transition  $\theta_{0\eta_1}$  is in  $E(K(Y)[G])$ . Therefore, we can find a divisor  $D' \subset Y$  with  $\eta_1$  not in  $D'$  such that  $V = Y - (D \cup D')$  is affine and

$\theta_{0\eta_1}$  comes from an element of  $E(\mathcal{O}_Y(V)[G])$ . We pick a lift  $\tilde{\theta}_{0\eta_1} \in \text{St}(\mathcal{O}_Y(V)[G])$ . Notice that it follows that  $f_{\eta_1} = \theta_{0\eta_1}^{-1} \cdot f_0$  is a basis of  $\mathcal{E}$  over  $V$ . Since  $f_{\eta_1}$  is also a basis over the completion of the local ring  $\hat{\mathcal{O}}_{\eta_1}$  of  $Y$  at  $\eta_1$ , we conclude that  $f_{\eta_1}$  is in fact a basis on a Zariski open  $W \subset Y$  that contains both  $V$  and  $\eta_1$ .

Now suppose that  $\eta_2 < \eta_1$  is away from the finite set of points of  $\bar{\eta}_1$  that do not belong to  $W$  and the singular points of  $D$  on all fibers of  $Y$ . Then  $\mathcal{O}_Y(V) \subset \hat{\mathcal{O}}_{\eta_2}[D^{-1}] = \hat{\mathcal{O}}_{\eta_2}[\eta_1^{-1}]$  and so both  $\theta_{0\eta_1}$  and  $\theta_{0\eta_2}$  can be viewed as elements of  $E(\hat{\mathcal{O}}_{\eta_2}[\eta_1^{-1}][G])$ .

Let us consider  $\theta_{\eta_1\eta_2}$  for such  $\eta_2 < \eta_1$ . Since  $\eta_2$  is in  $W$ , both  $f_{\eta_1}$  and  $f_{\eta_2}$  are bases at  $\eta_2$  and therefore

$$\theta_{\eta_1\eta_2} \in \text{GL}(\hat{\mathcal{O}}_{\eta_2}[G]) \cap E(\hat{\mathcal{O}}_{\eta_1\eta_2}[G]).$$

(i) If  $\eta_1$  is horizontal, since  $\eta_2$  is on  $W$ ,  $\eta_2$  does not lie on any vertical component of  $D$ . Therefore, the order of  $G$  is invertible in  $\hat{\mathcal{O}}_{\eta_2}$  and  $\text{SK}_1(\hat{\mathcal{O}}_{\eta_2}[G]) = (1)$  which gives  $\text{SL}(\hat{\mathcal{O}}_{\eta_2}[G]) = E(\hat{\mathcal{O}}_{\eta_2}[G])$ . Hence, in this case  $\theta_{\eta_1\eta_2}$  is in  $E(\hat{\mathcal{O}}_{\eta_2}[G]) \subset E(\hat{\mathcal{O}}_{\eta_2}[\eta_1^{-1}][G])$ .

(ii) We obtain the same conclusion, i.e  $\theta_{\eta_1\eta_2}$  is in  $E(\hat{\mathcal{O}}_{\eta_2}[G])$ , if  $\eta_1$  is vertical but is away from the prime divisors of the order of  $G$ .

(iii) Suppose now that  $\eta_1$  is a vertical fiber over such a prime divisor  $p$  of  $\#G$ . We will then check that  $\theta_{\eta_1\eta_2}$ , which by the above is in  $\text{SL}(\hat{\mathcal{O}}_{\eta_2}[G])$ , is actually in  $E(\hat{\mathcal{O}}_{\eta_2}[G])$ . For this, it is enough to check that the class  $[\theta_{\eta_1\eta_2}]$  in  $\text{SK}_1(\hat{\mathcal{O}}_{\eta_2}[G])$  is trivial. Since  $\theta_{\eta_1\eta_2}$  is in  $E(\hat{\mathcal{O}}_{\eta_1\eta_2}[G])$ , the image of  $[\theta_{\eta_1\eta_2}]$  under

$$(6.8) \quad \text{SK}_1(\hat{\mathcal{O}}_{\eta_2}[G]) \rightarrow \text{SK}_1(\hat{\mathcal{O}}_{\eta_1\eta_2}[G])$$

is trivial. Since  $\hat{\mathcal{O}}_{\eta_2} \simeq W(k)[[T]]$ ,  $\hat{\mathcal{O}}_{\eta_1\eta_2} \simeq W(k)\{\{T\}\}$  with  $k$  the residue field of  $\eta_2$ , by Corollary 2.13 (b), the map (6.8) is injective. Hence,  $[\theta_{\eta_1\eta_2}] = 1$ , and therefore  $\theta_{\eta_1\eta_2}$  is in  $E(\hat{\mathcal{O}}_{\eta_2}[G])$ .

To recap, we have that for  $\eta_1$  on  $D$  and for almost all  $\eta_2 < \eta_1$ ,

$$\theta_{0\eta_2} \in E(\hat{\mathcal{O}}_{\eta_2}[\eta_1^{-1}][G]), \quad \theta_{\eta_1\eta_2} \in E(\hat{\mathcal{O}}_{\eta_2}[G])$$

(with  $\theta_{\eta_1\eta_2} \in E(\hat{\mathcal{O}}_{\eta_2}[G])$  if  $\eta_1$  is horizontal or vertical away from  $\#G$ ).

For these (almost all)  $\eta_2 < \eta_1$ , pick lifts  $\tilde{\theta}_{0\eta_2} \in \text{St}(\hat{\mathcal{O}}_{\eta_2}[\eta_1^{-1}][G])$ , and  $\tilde{\theta}_{\eta_1\eta_2} \in \text{St}(\hat{\mathcal{O}}_{\eta_2}[G])$ , in addition to our lift  $\tilde{\theta}_{0\eta_1} \in \text{St}(\mathcal{O}_Y(V)[G])$ . All these lifts map to elements of  $\text{St}(\hat{\mathcal{O}}_{\eta_2}[\eta_1^{-1}][G])$ . Indeed, when  $\eta_1$  is vertical over  $p \nmid \#G$ ,  $E(\hat{\mathcal{O}}_{\eta_2}[G]) \subset \text{SL}(\hat{\mathcal{O}}_{\eta_2}[\eta_1^{-1}][G]) = E(\hat{\mathcal{O}}_{\eta_2}[\eta_1^{-1}][G])$  with the last equality following from Proposition 2.7 and Morita equivalence. For all the other finite set of  $\eta_2 < \eta_1$  pick lifts  $\tilde{\theta}_{\eta_i\eta_j}$  such that  $\tilde{\theta}_{0\eta_2} \in \text{St}(\hat{\mathcal{O}}_{\eta_2}[D^{-1}][G])$ ,  $\tilde{\theta}_{\eta_1\eta_2} \in \text{St}(\hat{\mathcal{O}}_{\eta_1\eta_2}[G])$ . Notice that for almost all  $\eta_2 < \eta_1$ , we have

$$(6.9) \quad z(\tilde{\theta})_{0,\eta_1,\eta_2} = \tilde{\theta}_{0\eta_2} \cdot (\tilde{\theta}_{\eta_1\eta_2})^{-1} \cdot (\tilde{\theta}_{0\eta_1})^{-1} \in K_2(\hat{\mathcal{O}}_{\eta_2}[\eta_1^{-1}][G]).$$

Now in view of (PK1) and (PK2) of Definition 2.2, (6.7) and (6.9) implies that the cocycle  $z(\tilde{\theta})$  given using these lifts  $\tilde{\theta}_{\eta_i\eta_j}$  of  $\theta_{\eta_i\eta_j}$  lies in the group

$$K'_2(\mathbb{A}_{Y,012}[G]) \cdot K_2(\mathbb{A}_{Y,12}[G])^\flat.$$

Finally, we want to compare  $z(\tilde{\theta})$  with  $z(\tilde{\lambda})$  which is given by lifts  $\tilde{\lambda}_{\eta_i\eta_j}$  as in the statement of (2). If  $\eta_1$  is not on  $D$  then  $z(\tilde{\theta})_{0,\eta_1,\eta_2} = z(\tilde{\lambda})_{0,\eta_1,\eta_2}$ . Suppose that  $\eta_1$  is on  $D$ . Pick a lift

$\tilde{\mu}_1$  of  $\mu_1 \in E(\widehat{\mathcal{O}}_{\eta_1}[G])$  to  $\text{St}(\widehat{\mathcal{O}}_{\eta_1}[G])$ . By Lemma 6.1 we have the cocycle identity, where for simplicity, we replace the subscripts  $\eta_1, \eta_2$  by 1 and 2:

$$(6.10) \quad z(\tilde{\theta})_{0,1,2} = z(\mu_1, \lambda_{12})^{-1} \cdot z(\tilde{\lambda})_{0,1,2} \cdot z(\lambda_{01}\mu_1^{-1}, \mu_1)$$

where we set (recall  $\theta_{01} = \lambda_{01} \cdot \mu_1^{-1}$ ,  $\theta_{12} = \mu_1 \cdot \lambda_{12}$ )

$$\begin{aligned} z(\mu_1, \lambda_{12}) &= \tilde{\theta}_{12} \cdot (\tilde{\lambda}_{12})^{-1} \cdot (\tilde{\mu}_1)^{-1} \\ z(\lambda_{01}\mu_1^{-1}, \mu_1) &= \tilde{\lambda}_{01} \cdot (\tilde{\mu}_1)^{-1} \cdot (\tilde{\theta}_{01})^{-1}. \end{aligned}$$

The first expression is in  $K_2(\widehat{\mathcal{O}}_{\eta_1\eta_2}[G])$  and the second in  $K_2(\widehat{\mathcal{O}}_{0\eta_1}[G])$ . Therefore, by the above, for almost  $\eta_2 < \eta_1$ ,  $\eta_1$  on  $D$ , the cocycle  $z(\tilde{\lambda})_{0,\eta_1,\eta_2}$  lies in

$$K_2(\widehat{\mathcal{O}}_{\eta_2}[\eta_1^{-1}][G]) \cdot K_2(\widehat{\mathcal{O}}_{\eta_1\eta_2}[G]) \cdot K_2(\widehat{\mathcal{O}}_{0\eta_1}[G]).$$

We can conclude that  $z(\tilde{\lambda})$  lies in the group

$$K'_2(\mathbb{A}_{Y,012}[G]) \cdot K_2(\mathbb{A}_{Y,12}[G])^b \cdot K'_2(\mathbb{A}_{Y,01}[G])^b.$$

as desired.  $\square$

6.c.2. Assume that the  $\mathcal{O}_Y[G]$ -bundle  $\mathcal{E}$  has elementary structure  $\{e_{\eta_i}^h\}_h$  with transition matrices  $\lambda_{\eta_i\eta_j}$ . Let the element

$$z(\tilde{\lambda}) := \prod_{(\eta_0, \eta_1, \eta_2)} \tilde{\lambda}_{\eta_0\eta_2} \cdot (\tilde{\lambda}_{\eta_0\eta_1})^{-1} \cdot (\tilde{\lambda}_{\eta_1\eta_2})^{-1} \in K'_2(\mathbb{A}_{Y,012}[G]) \cdot K_2(\mathbb{A}_{Y,12}[G])^b \cdot K'_2(\mathbb{A}_{Y,01}[G])^b.$$

be as in (2) of Proposition 6.4 above.

**Definition 6.5.** Assume that the  $\mathcal{O}_Y[G]$ -bundle  $\mathcal{E}$  comes equipped with an elementary structure as above. We define the adelic second Chern class  $c_2(\mathcal{E})$  of  $\mathcal{E}$  to be the class of  $z(\tilde{\lambda})$  in  $\text{CH}_{\mathbb{A}}^2(Y[G])$ . (Note that this is in fact an abuse of notation since this class actually depends on the choice of elementary structure of  $\mathcal{E}$ , and not just on  $\mathcal{E}$  alone.)

The following result implies that  $c_2(\mathcal{E})$  only depends on  $\mathcal{E}$  with its elementary structure and is independent of the choices involved in the definition.

**Theorem 6.6.** a) The class  $z(\tilde{\lambda})$  in  $\text{CH}_{\mathbb{A}}^2(Y[G])$  is independent of the choice of lifts  $\tilde{\lambda}_{\eta_i\eta_j}$  used in the definition of  $z$ .

b) Let  $\{\mu_{\eta_i}\}$  be an elementary base change and put  $\theta_{\eta_i\eta_j} = \mu_{\eta_i}\lambda_{\eta_i\eta_j}\mu_{\eta_j}^{-1}$  for the elementary transitions in the new basis. For any choice of lifts  $\tilde{\theta}_{\eta_i\eta_j}$  that satisfy the requirements of Proposition 6.4 (2) we have

$$z(\tilde{\lambda}) \cdot z(\tilde{\theta})^{-1} \in \prod_{0 \leq i < j \leq 2} K'_2(\mathbb{A}_{Y,ij}[G])^b.$$

As a result, the class of  $z(\tilde{\lambda})$  in  $\text{CH}_{\mathbb{A}}^2(Y[G])$  is unchanged by an elementary base change.

*Proof of Theorem 6.6 (a).* We suppose that we have two choices of lifts  $\tilde{\lambda}_{\eta_i\eta_j}, \tilde{\lambda}'_{\eta_i\eta_j}$ ; we then have

$$\begin{aligned} z(\tilde{\lambda})_{0,1,2} &= \tilde{\lambda}_{02}(\tilde{\lambda}_{12})^{-1}(\tilde{\lambda}_{01})^{-1} \\ z(\tilde{\lambda}')_{0,1,2} &= \tilde{\lambda}'_{02}(\tilde{\lambda}'_{12})^{-1}(\tilde{\lambda}'_{01})^{-1}. \end{aligned}$$

(For simplicity, here we drop the symbol  $\eta$  from the subscripts.) It then follows that

$$z(\tilde{\lambda}) \cdot z(\tilde{\lambda}')^{-1} = \tilde{\lambda}_{02}(\tilde{\lambda}_{12})^{-1}(\tilde{\lambda}_{01})^{-1} \cdot \tilde{\lambda}'_{01}\tilde{\lambda}'_{12}(\tilde{\lambda}'_{02})^{-1}.$$

It will suffice to prove that for each  $0 \leq i < j \leq 2$  we have:

$$(6.11) \quad \prod_{\eta_i \eta_j} (\tilde{\lambda}_{ij})^{-1} \tilde{\lambda}'_{ij} \in K'_2(\mathbb{A}_{Y,ij}[G])^\flat.$$

This easily follows from the definitions (see Definition 2.2) and Proposition 6.4. For example, consider  $i = 0, j = 1$ . In this case, we have  $\lambda_{\eta_0 \eta_1} \in E(\hat{\mathcal{O}}_{Y, \eta_1}[G])$  for almost all  $\eta_1$ ; hence for such  $\eta_1$  we have chosen  $\tilde{\lambda}_{\eta_0 \eta_1}, \tilde{\lambda}'_{\eta_0 \eta_1} \in \text{St}(\hat{\mathcal{O}}_{Y, \eta_1}[G])$  and therefore

$$\tilde{\lambda}_{\eta_0 \eta_1} \cdot (\tilde{\lambda}'_{\eta_0 \eta_1})^{-1} \in K_2(\hat{\mathcal{O}}_{Y, \eta_1}[G]).$$

*Proof of Theorem 6.6 (b).* By definition for each  $i$

$$(6.12) \quad \mu_{\eta_i} \in E(\hat{\mathcal{O}}_{Y, \eta_i}[G])$$

and so it certainly follows that  $\theta_{\eta_i \eta_j} \in E(\hat{\mathcal{O}}_{Y, \eta_i \eta_j}[G])$ , i.e.  $\{\theta_{\eta_i \eta_j}\}$  are elementary.

Our base change can be performed in three steps: in each step we alter the bases only at points in codimension 0, 1 and 2, by each of  $\mu_{\eta_0}, \mu_{\eta_2}$  or  $\mu_{\eta_1}$  respectively. For ease of notation we again write  $\mu_i$ , resp.  $\lambda_{ij}$ , for  $\mu_{\eta_i}$ , resp.  $\lambda_{\eta_i \eta_j}$  etc.

*Step 1.* Here we just change the base at  $\eta_0$  by  $\mu_{\eta_0}$ . The new transition matrices are  $\theta_{01} = \mu_0 \lambda_{01}$ ,  $\theta_{02} = \mu_0 \lambda_{02}$ ,  $\theta_{12} = \lambda_{12}$ . We choose lifts  $\tilde{\lambda}_{ij}, \tilde{\theta}_{ij}$  as in Proposition 6.4. We also choose a lifting  $\tilde{\mu}_0$  of  $\mu_0$  as follows: there is a divisor  $D_\mu \subset Y$  that contains all the vertical fibers at primes that divide the order of the group such that  $U = Y - D_\mu$  is affine and  $\mu_0$  lies in  $E(\hat{\mathcal{O}}_Y(U)[G])$ ; we pick a lift  $\tilde{\mu}_0$  in  $\text{St}(\hat{\mathcal{O}}_Y(U)[G])$ . As a result, for almost all  $\eta_1$ ,  $\tilde{\mu}_0$  maps to  $\text{St}(\hat{\mathcal{O}}_{Y, \eta_1}[G])$ , and for all  $\eta_2$ ,  $\tilde{\mu}_0$  maps to  $\text{St}(\hat{\mathcal{O}}_{Y, \eta_2}[D^{-1}][G])$ . By Lemma 6.1 applied to the subset  $\{\lambda_{01}, \lambda_{12} = \theta_{12}, \mu_0, \lambda_{02} = \lambda_{01} \lambda_{12}, \theta_{01} = \mu_0 \lambda_{01}, \theta_{02} = \mu_0 \lambda_{02} = \mu_0 \lambda_{01} \lambda_{12}\}$  of  $\text{St}(\hat{\mathcal{O}}_{Y, 0 \eta_1 \eta_2}[G])$  we have

$$(6.13) \quad \begin{aligned} z(\theta_{01}, \theta_{12}) &= z(\mu_0 \lambda_{01}, \lambda_{12}) = z(\mu_0, \lambda_{01})^{-1} z(\mu_0, \lambda_{01} \lambda_{12}) z(\lambda_{01}, \lambda_{12}) \\ &= z(\mu_0, \lambda_{01})^{-1} z(\mu_0, \lambda_{02}) z(\lambda_{01}, \lambda_{12}) \end{aligned}$$

where

$$\begin{aligned} z(\mu_0, \lambda_{01}) &= \widetilde{\mu_0 \lambda_{01}} \cdot (\tilde{\lambda}_{01})^{-1} (\tilde{\mu}_0)^{-1} \in K_2(\hat{\mathcal{O}}_{Y, \eta_1}[G]), \\ z(\mu_0, \lambda_{02}) &= \widetilde{\mu_0 \lambda_{02}} \cdot (\tilde{\lambda}_{02})^{-1} (\tilde{\mu}_0)^{-1} \in K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_2}[G]). \end{aligned}$$

Notice that from Proposition 6.4 (a) and our choice of  $\tilde{\mu}_0$ , we can see that  $\prod_{\eta_2} z(\mu_0, \lambda_{02})$  is in  $K'_2(\mathbb{A}_{Y, 02}[G])$ . Similarly,  $\prod_{\eta_1} z(\mu_0, \lambda_{01})$  is in  $K'_2(\mathbb{A}_{Y, 01}[G])$ . The result then follows since we have shown

$$z(\tilde{\lambda}) \cdot z(\tilde{\theta})^{-1} \in \prod_{0 \leq i < j \leq 2} K'_2(\mathbb{A}_{Y, ij}[G]).$$

*Step 2.* We now change the bases at  $\eta_1$  by  $\mu_{\eta_1}$ . The new transition matrices are given by  $\theta_{01} = \lambda_{01} \mu_1^{-1}$ ,  $\theta_{02} = \lambda_{02}$ ,  $\theta_{12} = \mu_1 \lambda_{12}$ . We choose a lift  $\tilde{\mu}_1 \in \text{St}(\hat{\mathcal{O}}_{Y, \eta_1}[G])$ . By Lemma 6.1 we have

$$(6.14) \quad z(\theta_{01}, \theta_{12}) = z(\lambda_{01} \mu_1^{-1}, \mu_1 \lambda_{12}) = z(\mu_1, \lambda_{12})^{-1} z(\lambda_{01}, \lambda_{12}) z(\lambda_{01} \mu_1^{-1}, \mu_1)$$



where

$$\begin{aligned} z(\mu_1, \lambda_{12}) &= \widetilde{\mu_1 \lambda_{12}} (\tilde{\lambda}_{12})^{-1} (\tilde{\mu}_1)^{-1} \in K_2(\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}[G]) \\ z(\lambda_{01} \mu_1^{-1}, \mu_1) &= \tilde{\lambda}_{01} (\tilde{\mu}_1)^{-1} (\widetilde{\lambda_{01} \mu_1^{-1}})^{-1} \in K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_1}[G]). \end{aligned}$$

By hypothesis,  $\lambda_{01}$  is in  $E(\hat{\mathcal{O}}_{Y, \eta_1}[D^{-1}][G])$ , and so  $\prod_{\eta_1} z(\lambda_{01} \mu_1^{-1}, \mu_1)$  is in  $K'_2(\mathbb{A}_{Y, 01}[G])$ . The result then follows.

*Step 3.* We now change the bases at  $\eta_2$  by  $\mu_{\eta_2}^{-1}$  (the inverse is for ease in the notation below). The new transition matrices are then given by  $\theta_{01} = \lambda_{01}$ ,  $\theta_{12} = \lambda_{12} \mu_2$ ,  $\theta_{02} = \lambda_{02} \mu_2$ . In addition to the lifts  $\tilde{\lambda}_{ij}$ ,  $\tilde{\theta}_{ij}$  we choose lifts  $\tilde{\mu}_{\eta_2} \in \text{St}(\hat{\mathcal{O}}_{Y, \eta_2}[G])$ . By Lemma 6.1 we have

$$\begin{aligned} (6.15) \quad z(\theta_{01}, \theta_{12}) &= z(\lambda_{01}, \lambda_{12} \mu_2) = z(\lambda_{12}, \mu_2)^{-1} z(\lambda_{01} \lambda_{12}, \mu_2) z(\lambda_{01}, \lambda_{12}) \\ &= z(\lambda_{12}, \mu_2)^{-1} z(\lambda_{02}, \mu_2) z(\lambda_{01}, \lambda_{12}) \end{aligned}$$

where

$$\begin{aligned} z(\lambda_{12}, \mu_2) &= \widetilde{\lambda_{12} \mu_2} (\tilde{\mu}_2)^{-1} (\tilde{\lambda}_{12})^{-1} \in K_2(\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}[G]) \\ z(\lambda_{02}, \mu_2) &= \widetilde{\lambda_{02} \mu_2} (\tilde{\mu}_2)^{-1} (\tilde{\lambda}_{02})^{-1} \in K_2(\hat{\mathcal{O}}_{Y, \eta_0 \eta_2}[G]). \end{aligned}$$

Since  $\tilde{\lambda}_{02}$  and  $\tilde{\theta}_{02} = \widetilde{\lambda_{02} \mu_2}$  are in  $\text{St}(\hat{\mathcal{O}}_{Y, \eta_2}[D^{-1}][G])$ , for some divisor  $D$ , the last expression contributes a term  $\prod_{\eta_2} z(\lambda_{02}, \mu_2)$  in  $K'_2(\mathbb{A}_{Y, 02}[G])$  and the result follows. This completes the proof of Theorem 6.6.  $\square$

## 7. EQUIVARIANT EULER CHARACTERISTICS AND THE RIEMANN-ROCH THEOREM

We can now state our main result concerning equivariant coherent Euler characteristics. We refer the reader to [6] or [7] for the construction of the projective equivariant Euler characteristic. See also the beginning of the introduction. Recall that we can identify the locally free class group  $\text{Cl}(\mathbb{Z}[G])$  with both the kernel  $K_0^{\text{red}}(\mathbb{Z}[G])$  of the rank map and with the quotient  $K_0(\mathbb{Z}[G])/\langle \mathbb{Z}[G] \rangle$ . We will denote by  $\bar{\chi}^P(Y, \mathcal{E})$  the image of the projective equivariant Euler characteristic  $\chi^P(Y, \mathcal{E})$  in  $\text{Cl}(\mathbb{Z}[G]) = K_0(\mathbb{Z}[G])/\langle \mathbb{Z}[G] \rangle$ .

**Theorem 7.1.** *Let  $Y$  be a regular flat projective scheme over  $\text{Spec}(\mathbb{Z})$  of dimension 2, with structure morphism  $h : Y \rightarrow S = \text{Spec}(\mathbb{Z})$ . Assume in addition that  $Y$  satisfies assumption (H) and that  $\mathbb{Q}[G]$  splits as in (0.1). Let  $\mathcal{E}$  be an  $\mathcal{O}_Y[G]$ -bundle which has an elementary structure in the sense of Definition 6.2. Then*

$$(7.1) \quad \bar{\chi}^P(Y, \mathcal{E}) = -h_*(c_2(\mathcal{E}))$$

$$\text{in } \text{Cl}(\mathbb{Z}[G]) = K_0^{\text{red}}(\mathbb{Z}[G]) = \text{CH}_{\mathbb{A}}^1(S[G]).$$

Suppose that  $\mathcal{E}$  has rank  $n$ . Since  $\mathcal{E}$  has elementary structure,  $c_1(\mathcal{E})$  is trivial and the usual Riemann-Roch theorem for the generic fiber  $Y_{\mathbb{Q}}$  shows that the rank of  $\chi^P(Y, \mathcal{E})$  is equal to that of  $\chi^P(Y, \mathcal{O}_Y[G]^n)$ . Since  $\chi^P(Y, \mathcal{O}_Y[G]^n)$  is the class of a free  $\mathbb{Z}[G]$ -module, the above formulation of the main result is equivalent to the one in the introduction. In §8 we reduce the proof of this result to the case  $Y = \mathbb{P}^1 = \mathbb{P}_{\mathbb{Z}}^1$ . When  $Y = \mathbb{P}^1$ , a stronger result is proved in §9.

8. THE PROOF OF THE THEOREM; REDUCTION TO THE CASE OF  $\mathbb{P}_{\mathbb{Z}}^1$ .

Throughout this section we suppose that  $h : Y \rightarrow \operatorname{Spec}(\mathbb{Z})$ ,  $G$  and  $\mathcal{E}$  are as in Theorem 7.1. By a result of B. Green (see [21] and [20] but also [8]), there is a finite flat morphism  $\pi : Y \rightarrow \mathbb{P}^1 = \mathbb{P}_{\mathbb{Z}}^1$ . Let  $f : \mathbb{P}^1 \rightarrow \operatorname{Spec}(\mathbb{Z})$  be the structure morphism, so that  $h = f \circ \pi$ . Let  $d$  be the degree of  $\pi$ . We can view  $\mathcal{V} = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^1}}(\pi_* \mathcal{O}_Y[G]^n, \mathcal{O}_{\mathbb{P}^1})$  and  $\mathcal{V}' = \pi_* \mathcal{O}_Y[G]^n$  as locally free  $\mathcal{O}_{\mathbb{P}^1}[G]$ -modules of rank  $nd$ . Parts (iii) and (iv) of the following result imply Theorem 7.1.

**Theorem 8.1.** *Let  $\mathcal{E}$  be an  $\mathcal{O}_Y[G]$ -bundle with elementary structure.*

- i) *The bundles  $\pi_*(\mathcal{E}) \oplus \mathcal{V}$  and  $\mathcal{V}' \oplus \mathcal{V}$  are locally free and have elementary structures as  $\mathcal{O}_{\mathbb{P}^1}[G]$ -bundles. They therefore have well defined second Chern classes  $c_2(\pi_* \mathcal{E} \oplus \mathcal{V})$  and  $c_2(\mathcal{V}' \oplus \mathcal{V})$  in  $\operatorname{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$ .*
- ii) *There is a push down map  $\pi_* : \operatorname{CH}_{\mathbb{A}}^2(Y[G]) \rightarrow \operatorname{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$  induced by  $\pi : Y \rightarrow \mathbb{P}^1$ . One has*

$$(8.1) \quad c_2(\pi_*(\mathcal{E}) \oplus \mathcal{V}) = \pi_*(c_2(\mathcal{E})) + c_2(\mathcal{V}' \oplus \mathcal{V}).$$

- iii) *There are equalities of equivariant Euler characteristics*

$$(8.2) \quad \bar{\chi}^P(Y, \mathcal{E}) = \bar{\chi}^P(\mathbb{P}^1, \pi_* \mathcal{E}) = \bar{\chi}^P(\mathbb{P}^1, \pi_* \mathcal{E} \oplus \mathcal{V})$$

$$\text{in } \operatorname{Cl}(\mathbb{Z}[G]) = \operatorname{K}_0^{\operatorname{red}}(\mathbb{Z}[G]) = \operatorname{CH}_{\mathbb{A}}^1(S[G]).$$

- iv) *We have*

$$(8.3) \quad \begin{aligned} \bar{\chi}^P(\mathbb{P}^1, \pi_* \mathcal{E} \oplus \mathcal{V}) &= -f_*(c_2(\pi_* \mathcal{E} \oplus \mathcal{V})) = -f_*(\pi_*(c_2(\mathcal{E})) + c_2(\mathcal{V}' \oplus \mathcal{V})) \\ &= -f_*(\pi_*(c_2(\mathcal{E}))) = -h_*(c_2(\mathcal{E})). \end{aligned}$$

Notice that, in view of (i), the first equality in part (iv) above follows from the case  $Y = \mathbb{P}^1$  of Theorem 7.1.

## 8.a. Constructing bundles with elementary structures.

**Lemma 8.2.** *Suppose  $R$  is an arbitrary ring. There is an order two automorphism  $\sigma$  of  $\operatorname{K}_1(R) = \operatorname{GL}(R)/\operatorname{E}(R) = \operatorname{GL}(R)^{\operatorname{ab}}$  induced by the anti-involution  $\sigma : A \rightarrow A^t$  on  $\operatorname{GL}(R)$ , where  $A^t$  is the transpose of the matrix  $A$ . This involution is trivial if and only if for all  $A \in \operatorname{GL}(R)$ , the block matrix*

$$(8.4) \quad \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$$

*lies in  $\operatorname{E}(R)$ . This is the case, in particular, if  $R$  is commutative and  $\operatorname{SK}_1(R)$  is trivial.*

*Proof.* By the Whitehead Lemma,  $\operatorname{E}(R)$  is the commutator subgroup of  $\operatorname{GL}(R)$ . If  $[A, B] = ABA^{-1}B^{-1}$  is a commutator, then  $\sigma([A, B]) = [(B^t)^{-1}, (A^{-1})^t]$  is also a commutator. Hence  $\operatorname{E}(R)$  is stable under  $\sigma$ . Since  $(AB)^t = B^t A^t$  for all  $A, B \in \operatorname{GL}(R)$  and  $\operatorname{K}_1(R)$  is the maximal abelian quotient of  $\operatorname{GL}(R)$ ,  $\sigma$  defines a group automorphism of  $\operatorname{K}_1(R)$ . By [46, Corollary 2.1.3], the block matrix

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$$

lies in  $E(R)$  for all  $A \in \mathrm{GL}(R)$ . Hence  $\sigma(A) = A^t$  equals  $A$  in  $K_1(R)$  if and only if (8.4) lies in  $E(R)$ . If  $A$  commutative and  $\mathrm{SK}_1(A)$  is trivial, then  $\sigma$  is trivial since  $\det(A) = \det(A^t)$ .  $\square$

We do not know whether  $\sigma$  is trivial for arbitrary  $R$ .

**Proposition 8.3.** *Let  $\mathcal{E}$  be as in Theorem 7.1. The direct image  $\pi_*\mathcal{E}$  is a rank  $n$  locally free sheaf of  $\pi_*\mathcal{O}_Y[G]$ -modules on  $\mathbb{P}^1 = \mathbb{P}_{\mathbb{Z}}^1$  as well as a locally free sheaf of  $\mathcal{O}_{\mathbb{P}^1}[G]$ -modules of rank  $nd$ . The sheaves  $\mathcal{V} = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^1}}(\pi_*\mathcal{O}_Y[G]^n, \mathcal{O}_{\mathbb{P}^1})$  and  $\mathcal{V}' = \pi_*\mathcal{O}_Y[G]^n$  are locally free  $\mathcal{O}_{\mathbb{P}^1}[G]$ -modules of rank  $nd$ .*

i) *There are equalities of equivariant Euler characteristics*

$$(8.5) \quad \bar{\chi}^P(Y, \mathcal{E}) = \bar{\chi}^P(\mathbb{P}^1, \pi_*\mathcal{E}) \quad \text{and} \quad \bar{\chi}^P(\mathbb{P}^1, \mathcal{V}) = \bar{\chi}^P(\mathbb{P}^1, \mathcal{V}') = 0.$$

in  $\mathrm{Cl}(\mathbb{Z}[G]) = K_0^{\mathrm{red}}(\mathbb{Z}[G]) = \mathrm{CH}_{\mathbb{A}}^1(S[G])$ .

ii) *The bundles  $\pi_*\mathcal{E} \oplus \mathcal{V}$  and  $\mathcal{V}' \oplus \mathcal{V}$  have elementary structures on  $\mathbb{P}^1$ . The restrictions of  $\pi_*\mathcal{E} \oplus \mathcal{V}$  and  $\mathcal{V}' \oplus \mathcal{V}$  to the zero section of  $\mathbb{P}^1$  define locally free  $\mathbb{Z}[G]$ -modules which are stably free.*

*Proof.* The first equality in (8.5) is clear. Since  $\mathcal{V}$  and  $\mathcal{V}'$  are induced from the trivial subgroup of  $G$ , they have trivial stable Euler characteristics as in (8.5). We now show (ii) for the bundle  $\pi_*\mathcal{E} \oplus \mathcal{V}$ , since the case of  $\mathcal{V}' \oplus \mathcal{V}$  is similar.

We will use  $\eta'_i$  to denote a point of  $Y$ . By assumption, there is a set of  $\hat{\mathcal{O}}_{Y, \eta'_i}[G]$  bases  $\{e_{\eta'_i}^h\}_h$  of  $\mathcal{E} \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{Y, \eta'_i}$  which has the properties of Definition 6.2 when one replaces  $\eta_i$  in this definition by  $\eta'_i$ . Here  $h$  runs from 1 to  $n = \mathrm{rank}_{\mathcal{O}_Y[G]}(\mathcal{E})$ . Suppose  $\eta$  is a non-degenerate Parshin chain on  $\mathbb{P}^1$ . Proposition 1.5 shows that

$$(8.6) \quad \hat{\mathcal{O}}_{\mathbb{P}^1, \eta} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \pi_*\mathcal{O}_Y[G] = \bigoplus_{\eta' \in \pi^{-1}(\eta)} \hat{\mathcal{O}}_{Y, \eta'}[G]$$

where  $\eta'$  runs over the Parshin chains on  $Y$  over  $\eta$ . We thus have

$$(8.7) \quad \pi_*\mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \hat{\mathcal{O}}_{\mathbb{P}^1, \eta} = \bigoplus_{\eta' \in \pi^{-1}(\eta)} \mathcal{E} \otimes_{\mathcal{O}_Y} \hat{\mathcal{O}}_{Y, \eta'}.$$

The bases  $\{e_{\eta'_i}^h\}_h$  together with the isomorphisms (8.6) and (8.7) give a set of local bases  $\{e_{\eta_i}^h\}$  for  $\pi_*\mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \hat{\mathcal{O}}_{\mathbb{P}^1, \eta_i}$  as a  $(\pi_*\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^1}} \hat{\mathcal{O}}_{\mathbb{P}^1, \eta_i})[G]$ -module, where  $\eta_i$  ranges over the points of  $\mathbb{P}^1$ .

We now consider transition matrices. Let  $\eta = (\eta_i, \eta_j)$  be a non-degenerate Parshin chain of length two on  $\mathbb{P}^1$ . We then have a transition map  $\lambda_\eta$  in  $\mathrm{GL}_n(\hat{\mathcal{O}}_{\mathbb{P}^1, \eta} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \pi_*\mathcal{O}_Y[G])$  determined by the bases  $\{e_{\eta_i}^h\}$  and  $\{e_{\eta_j}^h\}$  for the completion of  $\pi_*\mathcal{E}$  at  $\eta_i$  and  $\eta_j$ , respectively. The isomorphisms (8.6) and (8.7) identify  $\lambda_\eta$  with the direct sum of the transition matrices  $\lambda_{\eta'}$  which result from taking  $\{e_{\eta_i}^h\}$  (resp.  $\{e_{\eta_j}^h\}$ ) as a basis for the completion of  $\mathcal{E}$  at each point  $\eta'_i$  (resp.  $\eta'_j$ ) of  $Y$  over  $\eta_i$  (resp.  $\eta_j$ ).

We can choose a basis  $\{w_{\eta_i}^\ell\}_\ell$  for  $(\pi_*\mathcal{O}_Y)_{\eta_i}$  as a free module for  $\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_i}$  at each point  $\eta_i$  of  $\mathbb{P}^1$  which has the following properties. The index  $\ell$  runs from 1 to the degree  $d$  of  $\pi : Y \rightarrow \mathbb{P}^1$ . If  $\eta_0$  is the generic point of  $\mathbb{P}^1$ , then  $w_{\eta_i} = w_{\eta_0}$  for almost all codimension 1 points  $\eta_i$ . For each codimension 1 point  $\eta_i$ , we can arrange that for almost all closed points  $\eta_j$  on the closure of  $\eta_i$ , the basis element  $w_{\eta_j}^\ell$  equals  $w_{\eta_i}^\ell$ .

Returning now to our original set-up, we have obtained a basis  $W_{\eta_i} = \{w_{\eta_i}^\ell e_{\eta_i}^h\}_{\ell,h}$  for  $\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_i} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \pi_* \mathcal{E}$  as a free module for  $\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_i}[G]$ .

Recall  $\mathcal{V}' = \pi_* \mathcal{O}_Y[G]^n$ , which we may consider as either a  $\pi_* \mathcal{O}_Y[G]$ -module or as a  $\mathcal{O}_{\mathbb{P}^1}[G]$ -module. Let  $\{e'^h\}_h$  be a global basis for  $\mathcal{V}'$  as a free  $\pi_* \mathcal{O}_Y[G]$ -module of rank  $d$ . Then  $W'_{\eta_i} = \{w_{\eta_i}^\ell e'^h\}_{\ell,h}$  gives a basis for  $\mathcal{V}'_{\eta_i} = (\pi_* \mathcal{V}')_{\eta_i}$  as locally free  $\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_i}[G]$ -module of rank  $nd$ .

We use the bases  $W_{\eta_i}$ ,  $W_{\eta_j}$ ,  $W'_{\eta_i}$  and  $W'_{\eta_j}$  to arrive at transition matrices  $\lambda_{W, \eta}$  and  $\lambda_{W', \eta}$  in  $\mathrm{GL}_{nd}(\hat{\mathcal{O}}_{\mathbb{P}^1, \eta}[G])$  for  $\pi_* \mathcal{E}$  and  $\mathcal{V}'$  considered as locally free  $\mathcal{O}_{\mathbb{P}^1}[G]$ -modules. Note that  $\lambda_{W', \eta}$  lies in  $\mathrm{GL}_{nd}(\hat{\mathcal{O}}_{\mathbb{P}^1, \eta})$ , i.e. its entries have group ring elements which are in fact constants. To compare  $\lambda_{W, \eta}$  and  $\lambda_{W', \eta}$ , we will use the embedding

$$r_j : \mathrm{GL}_n(\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_j} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \pi_* \mathcal{O}_Y[G]) \rightarrow \mathrm{GL}_{nd}(\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_j}[G])$$

which results from the basis  $\{w_{\eta_j}^\ell\}_\ell$  for  $(\pi_* \hat{\mathcal{O}}_Y)_{\eta_j}$  as a free module for  $\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_j}$ . This extends by tensor product with  $\hat{\mathcal{O}}_{\mathbb{P}^1, \eta}$  to an embedding

$$(8.8) \quad r_j : \mathrm{GL}_n(\hat{\mathcal{O}}_{\mathbb{P}^1, \eta} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \pi_* \mathcal{O}_Y[G]) \rightarrow \mathrm{GL}_{nd}(\hat{\mathcal{O}}_{\mathbb{P}^1, \eta}[G])$$

where  $\eta = (\eta_i, \eta_j)$  as before. Composing  $r_j(\lambda_\eta)$  with the transition matrix  $\lambda_{W', \eta}$  associated with changing bases for  $\mathcal{V}'$  from the  $W'_{\eta_j}$  to  $W'_{\eta_i}$  gives the transition matrix  $\lambda_{W, \eta}$  associated with changing bases for the completion of  $\pi_* \mathcal{E}$  from  $W_{\eta_j}$  to  $W_{\eta_i}$ . We thus have the matrix equation

$$(8.9) \quad \lambda_{W, \eta} = \lambda_{W', \eta} \cdot r_j(\lambda_\eta)$$

inside  $\mathrm{GL}_{nd}(\hat{\mathcal{O}}_{\mathbb{P}^1, \eta}[G])$ .

As in the statement, let  $\mathcal{V}$  be the locally free  $\hat{\mathcal{O}}_{\mathbb{P}^1}[G]$ -module of rank  $nd$  defined by

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^1}}(\pi_* \mathcal{O}_Y[G]^n, \mathcal{O}_{\mathbb{P}^1}) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{V}', \mathcal{O}_{\mathbb{P}^1}).$$

Let  $W'^*_{\eta_i}$  be the basis for  $\mathcal{V}$  which is the  $\mathcal{O}_{\mathbb{P}^1}$  dual to the basis  $W'_{\eta_i}$  for  $\mathcal{V}'$  at  $\eta_i$ . Then the transition matrix  $\lambda_{W'^*, \eta}$  associated to this choice is

$$(8.10) \quad \lambda_{W'^*, \eta} = (\lambda_{W', \eta}^t)^{-1}$$

where the superscript  $t$  on the right stands for the transpose.

We conclude that the transition matrix for  $\pi_* \mathcal{E} \oplus \mathcal{V}'$  has the block form

$$(8.11) \quad \begin{pmatrix} \lambda_{W', \eta} \cdot r_j(\lambda_\eta) & 0 \\ 0 & (\lambda_{W', \eta}^t)^{-1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \cdot \begin{pmatrix} r_j(\lambda_\eta) & 0 \\ 0 & 1 \end{pmatrix}$$

where  $A = \lambda_{W', \eta}$ . To show  $\pi_* \mathcal{E} \oplus \mathcal{V}'$  has an elementary structure, it is enough by Definition 6.2 to show that each of the two matrices on the right side of (8.11) is elementary. The first matrix is elementary by Lemma 8.2 since the entries of  $A$  lie in the commutative local ring  $\hat{\mathcal{O}}_{\mathbb{P}^1, \eta}$  and  $\mathrm{SK}_1(\hat{\mathcal{O}}_{\mathbb{P}^1, \eta}) = \{1\}$  for all Parshin pairs  $\eta$  by Corollary 2.8. The second matrix on the right hand side of (8.11) is elementary because  $\lambda_\eta$  is so by assumption and  $r_j$  is a ring homomorphism.

The last statement to prove is that the restriction of  $\pi_* \mathcal{E} \oplus \mathcal{V}'$  to the zero section of  $\mathbb{P}^1$  is a stably free projective  $\mathbb{Z}[G]$ -module. This restriction has an elementary structure as a

projective  $\mathbb{Z}[G]$ -module. Therefore, its transition matrices have trivial determinant  $\text{Det}$  and hence this module is stably free by resolvent theory in dimension 1 (e.g [17]).  $\square$

**8.b. Steinberg extensions over  $Y$  and over  $\mathbb{P}^1$ .** Let  $(\eta_i, \eta_j)$  stand for a Parshin chain of length two on  $\mathbb{P}^1$ . For simplicity, we will denote by  $R_{ij}$ , resp.  $S_{ij}$ , the ring  $\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_i \eta_j}$ , resp.  $\hat{\mathcal{O}}_{Y, \eta_i \eta_j} = \pi_* \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^1}} \hat{\mathcal{O}}_{\mathbb{P}^1, \eta_i \eta_j}$ . A choice of basis of  $S_{ij}$  over  $R_{ij}$  yields a homomorphism

$$r : \text{GL}_n(S_{ij}[G]) \rightarrow \text{GL}_{nd}(R_{ij}[G])$$

which induces a map on elementary matrices

$$r_E : E_n(S_{ij}[G]) \rightarrow E_{nd}(R_{ij}[G])$$

and we get a diagram

$$(8.12) \quad \begin{array}{ccccccc} 1 & \rightarrow & K_2(S_{ij}[G]) & \rightarrow & \text{St}(S_{ij}[G]) & \rightarrow & E(S_{ij}[G]) \rightarrow 1 \\ & & \downarrow r_K & & \downarrow r_S & & \downarrow r_E \\ 1 & \rightarrow & K_2(R_{ij}[G]) & \rightarrow & \text{St}(R_{ij}[G]) & \rightarrow & E(R_{ij}[G]) \rightarrow 1. \end{array}$$

Suppose now that we change bases by a matrix  $c \in E(R_{ij})$  and we then replace  $r_E$  by  $r_E^c = c^{-1} r_E c$ ; since  $c$  elementary it has a lift  $s(c) \in \text{St}(R_{ij})$  and conjugation by this element is independent of the lift, as two such lifts differ by an element of  $K_2(R_{ij})$  which is central; finally, conjugation by  $s(c)$  on  $K_2(R_{ij}[G])$  is trivial and so  $r_K^{s(c)} = r_K$ .

More generally, reasoning as above, if we choose  $c \in E(R_{012}[G])$  (in fact  $c \in E(R_{012})$  will suffice for our purposes) and consider the effect of conjugation by  $c$ , denoted when necessary  $\text{conj}(c)$ , then we get a diagram:

$$(8.13) \quad \begin{array}{ccccccc} 1 & \rightarrow & K_2(R_{ij}[G]) & \rightarrow & \text{St}(R_{ij}[G]) & \xrightarrow[\pi_{ij}]{s_{ij}} & E(R_{ij}[G]) \rightarrow 1 \\ & & \downarrow \text{conj}(c) & & \downarrow \text{conj}(c) & & \downarrow \text{conj}(c) \\ 1 & \rightarrow & K_2(R_{ij}[G])^c & \rightarrow & \text{St}(R_{ij}[G])^c & \xrightarrow[\pi_{ij,c}]{s_{ij,c}} & E(R_{ij}[G])^c \rightarrow 1. \end{array}$$

Here  $s_{ij}$  and  $s_{ij,c}$  denote sections and  $K_2(R_{ij}[G])^c$ ,  $\text{St}(R_{ij}[G])^c$  are just formal copies of  $K_2(R_{ij}[G])$ ,  $\text{St}(R_{ij}[G])$ , while  $E(R_{ij}[G])^c$  is the conjugate in  $E(R_{012}[G])$ . Note here that for the images in  $K_2(R_{012}[G])$  we have  $(K_2(R_{ij}[G])^c)^b = (K_2(R_{ij}[G])^b)^c = K_2(R_{ij}[G])^b$ . Note also that the right-hand square commutes with respect to the  $\pi$  maps, but not necessarily for the section maps: to be more precise we have:

**Lemma 8.4.** *Given  $x \in E(R_{ij}[G])$  we have  $\kappa = s_{ij,c}(x^c) s_{ij}(x)^{-c} \in K_2(R_{ij}[G])^c$ .*

*Proof.* By definition  $\kappa$  is the product of two elements in  $\text{St}(R_{ij}[G])^c$  and it will suffice to show that  $\kappa \in \ker(\pi_{ij,c})$ . To this end we note:

$$\begin{aligned} \pi_{ij,c}(\kappa) &= \pi_{ij,c}(s_{ij,c}(x^c)) \cdot \pi_{ij,c}(s_{ij}(x)^{-c}) \\ &= \pi_{ij,c}(s_{ij,c}(x^c)) \cdot \pi_{ij}(s_{ij}(x))^{-c} \\ &= x^c \cdot x^{-c} = 1. \end{aligned}$$

$\square$

8.c. **Base change.** Let  $\{e_i\}$  be a basis for  $\mathcal{E} \otimes_{\mathcal{O}_Y} S_i$  over  $S_i[G]$ . (There is an additional implicit subscript we will suppress which runs from 1 to the rank of  $\mathcal{E}$ .) Let  $e_i = \mu_{ij} e_j$ .

Let  $\{a_{in}\}_{n=1}^d$  be a basis for  $S_i$  over  $R_i$ ; we let  $(a_{in})_n = \Lambda_{ij}(a_{jn})_n$  and, as in the proof of Proposition 8.3, we write

$$(8.14) \quad \Lambda_{ij}^\# = \begin{pmatrix} \Lambda_{ij} & 0 \\ 0 & (\Lambda_{ij}^t)^{-1} \end{pmatrix}.$$

We then have bases  $(a_{in}e_i)_n$  for  $\mathcal{E} \otimes_{\mathcal{O}_Y} S_i$  over  $R_i[G]$  and hence a further set of transition matrices

$$(8.15) \quad (a_{in}e_i)_n = \lambda_{ij} \cdot (a_{jn}e_j)_n$$

so that the bundle  $\pi_* \mathcal{E} \oplus \mathcal{V}$  has transition matrices

$$\lambda_{ij}^\# = \begin{pmatrix} \lambda_{ij} & 0 \\ 0 & (\Lambda_{ij}^t)^{-1} \end{pmatrix}.$$

We observe that by the very definition of the map  $r_i : \mathrm{GL}_n(S_i[G]) \rightarrow \mathrm{GL}_{nd}(R_i[G])$  we have the further equality:

$$(8.16) \quad (a_{in}e_i)_n = r_i(\mu_{ij}) \cdot (a_{in}e_j)_n.$$

Next we observe that by definition of  $r_k$

$$r_k(\mu_{ij}) \cdot (a_{kn}e_j)_n = (a_{kn}e_i)_n$$

while

$$\Lambda_{ki} \lambda_{ij} \Lambda_{jk} \cdot (a_{kn}e_j)_n = \Lambda_{ki} \lambda_{ij} \cdot (a_{jn}e_j)_n = \Lambda_{ki} \cdot (a_{in}e_i)_n = (a_{kn}e_i)_n.$$

Therefore, we deduce that

$$(8.17) \quad r_k(\mu_{ij}) = \Lambda_{ki} \lambda_{ij} \Lambda_{jk}, \quad r_k^\#(\mu_{ij}) = \Lambda_{ki}^\# \lambda_{ij}^\# \Lambda_{jk}^\#$$

where

$$r_k^\#(\mu_{ij}) = \begin{pmatrix} r_k(\mu_{ij}) & 0 \\ 0 & 1 \end{pmatrix};$$

and hence

$$(8.18) \quad \lambda_{ij} = \Lambda_{ik} r_k(\mu_{ij}) \Lambda_{kj}, \quad \lambda_{ij}^\# = \Lambda_{ik}^\# r_k^\#(\mu_{ij}) \Lambda_{kj}^\#.$$

From (8.17) we conclude that for all  $k, h$  one has

$$(8.19) \quad \Lambda_{ik}^\# r_k^\#(\mu_{ij}) \Lambda_{kj}^\# = \lambda_{ij}^\# = \Lambda_{ih}^\# r_h^\#(\mu_{ij}) \Lambda_{hj}^\#$$

and so

$$(8.20) \quad r_k^\#(\mu_{ij}) = \Lambda_{ki}^\# \Lambda_{ih}^\# r_h^\#(\mu_{ij}) \Lambda_{hj}^\# \Lambda_{jk}^\# = \Lambda_{kh}^\# r_h^\#(\mu_{ij}) \Lambda_{hk}^\#.$$

#### 8.d. Reduction step outline.

**Lemma 8.5.** *Assume the notation and hypotheses of Theorem 7.1 and Proposition 8.3. The finite flat morphism  $\pi : Y \rightarrow \mathbb{P}^1 = \mathbb{P}_{\mathbb{Z}}^1$  induces a pushdown homomorphism*

$$(8.21) \quad \pi_* : \mathrm{CH}_{\mathbb{A}}^2(Y[G]) \rightarrow \mathrm{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G]).$$

*The composition of the morphism  $\pi : Y \rightarrow \mathbb{P}^1$  with the structure morphism  $f : \mathbb{P}^1 \rightarrow \mathrm{Spec}(\mathbb{Z})$  gives the structure morphism  $h = f \circ \pi : Y \rightarrow \mathrm{Spec}(\mathbb{Z})$ . There is an equality of pushdown homomorphisms*

$$h_* = f_* \circ \pi_*$$

*from  $\mathrm{CH}_{\mathbb{A}}^2(Y[G])$  to  $\mathrm{Cl}(\mathbb{Z}[G]) = K_0^{\mathrm{red}}(\mathbb{Z}[G]) = \mathrm{CH}_{\mathbb{A}}^1(\mathrm{Spec}(\mathbb{Z})[G])$ .*

*Proof.* Setting  $i = 0$ , we fix a basis for the local ring  $\mathcal{O}_{Y, \eta'_0} = \pi_* \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1, \eta_0}$  at the generic point  $\eta'_0$  of  $Y$  as a free rank  $d$  module over  $\mathcal{O}_{\mathbb{P}^1, \eta_0}$  when  $\eta_0$  is the generic point of  $\mathbb{P}^1$ . This basis defines an algebra map  $r_0 : \hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2}[G] \rightarrow M_d(\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_0 \eta_1 \eta_2}[G])$  for every Parshin triple  $\eta_0 \eta_1 \eta_2$  on  $\mathbb{P}^1$ , where  $\hat{\mathcal{O}}_{Y, \eta_0 \eta_1 \eta_2} = \pi_* \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^1}} \hat{\mathcal{O}}_{\mathbb{P}^1, \eta_0 \eta_1 \eta_2}$  is the direct sum of  $\hat{\mathcal{O}}_{Y, \eta'_0 \eta'_1 \eta'_2}$  over all Parshin triples  $\eta'_0 \eta'_1 \eta'_2$  on  $Y$  lying over  $\eta_0 \eta_1 \eta_2$ . This  $r_0$  gives a diagram

$$(8.22) \quad \begin{array}{ccccccc} 1 & \rightarrow & K_2(S_{012}[G]) & \rightarrow & \mathrm{St}(S_{012}[G]) & \rightarrow & E(S_{012}[G]) \rightarrow 1 \\ & & \downarrow r_0 & & \downarrow r_0 & & \downarrow r_0 \\ 1 & \rightarrow & K_2(R_{012}[G]) & \rightarrow & \mathrm{St}(R_{012}[G]) & \rightarrow & E(R_{012}[G]) \rightarrow 1 \end{array}$$

for every Parshin triple  $\eta_0 \eta_1 \eta_2 = 012$  on  $\mathbb{P}^1$ , using a notation parallel to that in §8.b. We wish to show that we can define the map  $\pi_*$  in (8.21) by applying  $r_0$  to every local component of an element of  $\mathrm{CH}_{\mathbb{A}}^2(Y[G])$ . To show that this is well-defined we have to show that  $r_0$  takes the numerators to numerators and denominators to denominators in the definition of  $\mathrm{CH}_{\mathbb{A}}^2(Y[G])$  and  $\mathrm{CH}_{\mathbb{A}}^1(\mathbb{P}^1[G])$  in Definition 2.4.

Suppose  $x$  lies in the numerator of  $\mathrm{CH}_{\mathbb{A}}^2(Y[G])$  and that  $\eta'$  is a Parshin triple on  $\mathbb{P}^1$  which involves  $\eta_0$  and another point  $\eta_k$  of  $\mathbb{P}^1$ . Suppose that in computing the component  $r_0(x)'_{\eta'}$  of  $r_0(x)$  we replace the basis for  $\mathcal{O}_{Y, \eta'_0}$  as a module for  $\mathcal{O}_{\mathbb{P}^1, \eta_0}$  by a basis for  $\hat{\mathcal{O}}_{Y, \eta_k} = \pi_* \mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^1}} \hat{\mathcal{O}}_{\mathbb{P}^1, \eta_k}$  as a module for  $\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_k}$ . This changes the algebra homomorphism  $r_0$  to the algebra homomorphism  $r_k$  which results from conjugating  $r_0$  by the transition matrix  $\lambda_{W', \eta_0 \eta_k}$ . This matrix need not be elementary, but it does have constant coefficients. Let  $\lambda_{W', \eta_0 \eta_k}^{\dagger}$  be the matrix having a block in the upper left corner equal to  $\lambda_{W', \eta_0 \eta_k}$  and a one-by-one matrix block in the lower right corner with entry  $\det(\lambda_{W', \eta_0 \eta_k})^{-1}$ . Then  $\lambda_{W', \eta_0 \eta_k}^{\dagger}$  also conjugates  $r_0$  to  $r_k$ . Since  $\lambda_{W', \eta_0 \eta_k}^{\dagger}$  has determinant 1 and has coefficients in  $R_{0k}$ ,  $\lambda_{W', \eta_0 \eta_k}^{\dagger}$  is elementary by Corollary 2.8. Since  $\mathrm{St}(R_{012}[G])$  is a central extension of  $E(R_{012}[G])$  by  $K_2(R_{012}[G])$ , the conjugation action of a lift of  $\lambda_{W', \eta_0 \eta_k}^{\dagger}$  to  $\mathrm{St}(R_{012}[G])$  does not depend on the choice of this lift. Furthermore, this conjugation action is trivial on  $K_2(R_{012}[G])$ . We conclude that in the above recipe for computing  $\pi_*$ , we are free to replace  $r_0$  by  $r_k$  when computing components at Parshin chains  $\eta$  of  $\mathbb{P}^1$  which involve  $\eta_k$ .

The first step in showing that  $x \rightarrow r_0(x)$  gives a well-defined homomorphism  $\pi_*$  on second adelic Chow groups is to show that if  $x \in K'_2(\mathbb{A}_{Y, 012}[G])$ , then  $r_0(x) \in K'_2(\mathbb{A}_{\mathbb{P}^1, 012}[G])$ . From Definition 2.2, we see that this assertion amounts to saying that if  $x$  satisfies conditions



(PK1) and (PK2) then  $r_0(x)$  satisfies these conditions when  $Y$  is replaced by  $\mathbb{P}^1$ . Consider condition (PK1). For all but finitely many codimension 1 points  $\eta_1$  on  $\mathbb{P}^1$ , the component  $x_{\eta'_0 \eta'_1 \eta'_2}$  of  $x$  at each Parshin triple of  $Y$  for which  $\eta'_1$  lies above  $\eta_1$  will satisfy the condition in (PK1), namely

$$(8.23) \quad x_{\eta'_0 \eta'_1 \eta'_2} \in K_2(\hat{\mathcal{O}}_{Y, \eta'_1 \eta'_2}[G])^\flat$$

for all  $\eta'_2 \in \overline{\eta'_1}$ . In determining the component  $r_0(x)_{\eta_0 \eta_1 \eta_2}$  we are free to replace  $r_0$  by the homomorphism  $r_1$  defined above using local bases at the point  $\eta_1$ . Since  $r_1$  comes from an algebra homomorphism  $\hat{\mathcal{O}}_{Y, \eta_1 \eta_2}[G] \rightarrow M_d(\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_1 \eta_2}[G])$ , we see that (8.23) implies  $r_0(x)$  satisfies (PK1) for  $\mathbb{P}^1$ . The remaining assertions one must prove in order to show  $\pi_*$  is a well defined homomorphism from  $\mathrm{CH}_{\mathbb{A}}^2(Y[G])$  to  $\mathrm{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$  can be proved in a similar way.

Let  $\eta = (\eta_0, \eta_1, \eta_2)$  be a (non-degenerate) Parshin triple on  $\mathbb{P}^1$ . Then  $L = \hat{\mathcal{O}}_{\mathbb{P}^1, \eta}$  is the total fractions of the product of discrete valuation rings  $R = \hat{\mathcal{O}}_{\mathbb{P}^1, \eta_1 \eta_2}$ . We suppose that  $\eta_2$  has residue characteristic  $p$ . Then  $N = L \otimes_{\mathcal{O}_{\mathbb{P}^1}} \pi_* \mathcal{O}_Y$  is a product of the fields given by the multicompletions of  $\mathcal{O}_Y$  at the Parshin triples of  $Y$  lying over  $\eta$ . We must show that there is a commutative diagram

$$(8.24) \quad \begin{array}{ccc} K_2(N[G]) & \xrightarrow{\pi_*} & K_2(L[G]) \\ & \searrow (f \circ \pi)_* & \downarrow f_* \\ & & K_1(\mathbb{Q}_p[G]). \end{array}$$

Since  $N$  and  $L$  are products of fields of characteristic 0, we can reduce to the case in which  $G$  is the trivial group by Morita equivalence. There are now two cases to consider, both of which are dealt with by [30]. If  $\eta_1$  is horizontal, then  $f_*$  and  $(f \circ \pi)_*$  are tame symbols, and (8.24) is commutative by [30, Lemma 3]. If  $\eta_1$  is vertical, then  $f_*$  and  $(f \circ \pi)_*$  are Kato's residue maps, and the result we need is shown on page 160 of [30], four lines above Proposition 3. In fact, in this case, the res map for  $N$  is constructed from the res map on  $L$  via the norm map  $\pi_*$ .  $\square$

**Proposition 8.6.** *Assume the notation and hypotheses of Proposition 8.3 and Lemma 8.5 above. Then we have*

$$(8.25) \quad \pi_*(c_2(\mathcal{E})) + c_2(\mathcal{V}' \oplus \mathcal{V}) = c_2(\pi_*(\mathcal{E}) \oplus \mathcal{V}).$$

The proof of this result will be completed in §8.e.

8.d.1. We now summarise how the results proved thus far will reduce the proof of Theorem 7.1 to the case of  $Y = \mathbb{P}_{\mathbb{Z}}^1$ .

Part (i) of Theorem 8.1 follows from Proposition 8.3 (ii). Part (ii) of Theorem 8.1 is shown by Lemma 8.5 and Proposition 8.6. The equalities in part (iii) of Theorem 8.1 follow from Proposition 8.3 (i). The equalities in part (iv) will be shown in Theorem 9.2 of the next section using Proposition 8.3 (ii) to show that hypothesis (b) of Theorem 9.2 can be stably satisfied for  $\mathcal{F} = \pi_* \mathcal{E} \oplus \mathcal{V}$  and  $\mathcal{F} = \mathcal{V}' \oplus \mathcal{V}$ .

**8.e. Proof of Proposition 8.6.** In this section we will prove Proposition 8.6 via cocycle calculations. These calculations will require repeated use of Lemma 6.1, and in this Lemma one must make choices of various lifts in order for the identity in the Lemma to apply. We start by forming adelic cocycles that give representatives for the classes  $c_2(\pi_*\mathcal{E} \oplus \mathcal{V})$ ,  $\pi_*(c_2(\mathcal{E}))$  and  $c_2(\mathcal{V}' \oplus \mathcal{V})$  in  $\mathrm{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$ . These will be given by choosing lifts to suitable Steinberg groups of elementary transition matrices which are as in Proposition 6.4. These lifts have to satisfy the conditions of this Proposition 6.4 (2) with respect to some divisor  $\Delta$  in  $\mathbb{P}^1$  which contains the fibers over the primes dividing the order of  $G$ . We will call such lifts *acceptable*. By Theorem 6.6, we can calculate using any set of acceptable lifts.

To make the notation more clear we will use  $s_{i_1 \dots i_k}(\lambda)$  instead of  $\tilde{\lambda}$  for a lift to the Steinberg group associated to a Parshin chain  $\eta_{i_1} \dots \eta_{i_k}$  of an elementary matrix  $\lambda$ . (Although the notation might be suggesting this, we are not choosing sections of the Steinberg sequence.) Recall that we have elementary transition matrices  $\lambda_{ij}^\sharp$  for the  $\mathcal{O}_{\mathbb{P}^1}[G]$ -bundle  $\pi_*\mathcal{E} \oplus \mathcal{V}$ . By Proposition 6.4 we can choose acceptable lifts:

A)  $s_{01}(\lambda_{01}^\sharp)$ ,  $s_{02}(\lambda_{02}^\sharp)$  of  $\lambda_{01}^\sharp$ ,  $\lambda_{02}^\sharp$ .

Similarly, for  $\mathcal{V}' \oplus \mathcal{V}$  and its transitions we can choose acceptable lifts

B)  $s_{ij}(\Lambda_{ij}^\sharp)$  of  $\Lambda_{ij}^\sharp$ .

We can also choose acceptable lifts

C)  $s_{01}(r_0^\sharp(\mu_{01}))$ ,  $s_{12}(r_2^\sharp(\mu_{12}))$ ,  $s_{02}(r_0^\sharp(\mu_{02}))$  of the matrices  $r_0^\sharp(\mu_{01})$ ,  $r_2^\sharp(\mu_{12})$ ,  $r_0^\sharp(\mu_{02})$ .

(These last three matrices in (C) are integral in the sense of Proposition 6.4 (1) with respect to some divisor  $\Delta$  in  $\mathbb{P}^1$  which contains the fibers over the primes dividing the order of  $G$ . This follows since  $\mu_{ij}$  are elementary transition matrices for the  $\mathcal{O}_Y[G]$ -bundle and so they satisfy the conclusion of Proposition 6.4 (1) over  $Y$  for a divisor  $D$  on  $Y$ . Indeed, it is now enough to take any  $\Delta$  that contains the image of  $D$  under  $\pi : Y \rightarrow \mathbb{P}^1$  together with the complement of the open of  $\mathbb{P}^1$  where the generic basis of  $\pi_*\mathcal{O}_Y$  involved in the choice of  $r_0$  is actually a basis.)

Starting from these lifts we now also consider lifts of some additional elements as follows:

D) We lift  $\lambda_{12}^\sharp = \Lambda_{12}^\sharp \cdot r_2^\sharp(\mu_{12})$  by setting

$$(8.26) \quad s_{12}(\lambda_{12}^\sharp) := s_{12}(\Lambda_{12}^\sharp) \cdot s_{12}(r_2^\sharp(\mu_{12})).$$

E) We lift  $r_0^\sharp(\mu_{12}) = \Lambda_{02}^\sharp \cdot r_2^\sharp(\mu_{12}) \cdot (\Lambda_{02}^\sharp)^{-1}$  by setting

$$(8.27) \quad s_{012}(r_0^\sharp(\mu_{12})) := s_{02}(\Lambda_{02}^\sharp) \cdot s_{12}(r_2^\sharp(\mu_{12})) \cdot s_{02}(\Lambda_{02}^\sharp)^{-1}.$$

F) We lift  $\Lambda_{01}^\sharp \cdot \lambda_{12}^\sharp = r_0^\sharp(\mu_{12}) \cdot \Lambda_{02}^\sharp = \Lambda_{02}^\sharp \cdot r_2^\sharp(\mu_{12})$  (see (8.18)) by setting

$$(8.28) \quad s_{012}(\Lambda_{01}^\sharp \cdot \lambda_{12}^\sharp) = s_{012}(r_0^\sharp(\mu_{12}) \cdot \Lambda_{02}^\sharp) = s_{012}(\Lambda_{02}^\sharp \cdot r_2^\sharp(\mu_{12})) := s_{02}(\Lambda_{02}^\sharp) \cdot s_{12}(r_2^\sharp(\mu_{12})).$$

Using these lifts, we can now calculate our various cocycles. We denote by  $z(\lambda^\sharp)$  the adelic element with components

$$(8.29) \quad z(\lambda^\sharp)_{0,1,2} = z(\lambda_{01}^\sharp, \lambda_{12}^\sharp) := s_{02}(\lambda_{02}^\sharp) \cdot s_{12}(\lambda_{12}^\sharp)^{-1} \cdot s_{01}(\lambda_{01}^\sharp)^{-1}.$$

Similarly consider  $z(\Lambda^\sharp)$  to be the adelic element with components

$$(8.30) \quad z(\Lambda^\sharp)_{0,1,2} = z(\Lambda_{01}^\sharp, \Lambda_{12}^\sharp) := s_{02}(\Lambda_{02}^\sharp) \cdot s_{12}(\Lambda_{12}^\sharp)^{-1} \cdot s_{01}(\Lambda_{01}^\sharp)^{-1}.$$

Also consider the adelic element  $z(r_0^\sharp(\mu))$  with components

$$(8.31) \quad z(r_0^\sharp(\mu))_{0,1,2} = z(r_0^\sharp(\mu_{01}), r_0^\sharp(\mu_{12})) := s_{02}(r_0^\sharp(\mu_{02})) \cdot s_{012}(r_0^\sharp(\mu_{12}))^{-1} \cdot s_{01}(r_0^\sharp(\mu_{01}))^{-1}.$$

The class of  $z(\lambda^\sharp)$  in  $\mathrm{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$  corresponds to  $c_2(\pi_*\mathcal{E} \oplus \mathcal{V})$ . By the construction in Lemma 8.5, the class of  $z(r_0^\sharp(\mu))$  corresponds to  $\pi_*c_2(\mathcal{E})$ . Finally, the class of  $z(\Lambda^\sharp)$  corresponds to  $c_2(\mathcal{V}' \oplus \mathcal{V})$ . Thus Proposition 8.6 will follow if we can show that

$$(8.32) \quad z(\lambda^\sharp) \cdot z(r_0^\sharp(\mu))^{-1} \cdot z(\Lambda^\sharp)^{-1} \in \prod_{0 \leq i < j \leq 2} K'_2(\mathbb{A}_{\mathbb{P}^1, ij}[G])^\flat.$$

**Lemma 8.7.** *We have an equality*

$$(8.33) \quad \begin{aligned} z(\lambda_{01}^\sharp, \lambda_{12}^\sharp) z(r_0^\sharp(\mu_{01}), \Lambda_{01}^\sharp) &= z(r_0^\sharp(\mu_{01}) \Lambda_{01}^\sharp, \lambda_{12}^\sharp) z(r_0^\sharp(\mu_{01}), \Lambda_{01}^\sharp) \\ &= z(r_0^\sharp(\mu_{01}), \Lambda_{01}^\sharp \lambda_{12}^\sharp) z(\Lambda_{01}^\sharp, \lambda_{12}^\sharp) \end{aligned}$$

where

$$\begin{aligned} z(\lambda_{01}^\sharp, \lambda_{12}^\sharp) &= z(r_0^\sharp(\mu_{01}) \Lambda_{01}^\sharp, \lambda_{12}^\sharp) = s_{02}(\lambda_{02}^\sharp) s_{12}(\lambda_{12}^\sharp)^{-1} s_{01}(\lambda_{01}^\sharp)^{-1} \\ z(r_0^\sharp(\mu_{01}), \Lambda_{01}^\sharp) &= s_{01}(\lambda_{01}^\sharp) s_{01}(\Lambda_{01}^\sharp)^{-1} s_{01}(r_0^\sharp(\mu_{01}))^{-1} \\ z(r_0^\sharp(\mu_{01}), \Lambda_{01}^\sharp \lambda_{12}^\sharp) &= s_{02}(\lambda_{02}^\sharp) s_{012}(\Lambda_{01}^\sharp \lambda_{12}^\sharp)^{-1} s_{01}(r_0^\sharp(\mu_{01}))^{-1} \\ z(\Lambda_{01}^\sharp, \lambda_{12}^\sharp) &= s_{012}(\Lambda_{01}^\sharp \lambda_{12}^\sharp) s_{12}(\lambda_{12}^\sharp)^{-1} s_{01}(\Lambda_{01}^\sharp)^{-1} \end{aligned}$$

with all lifts as defined in (A)-(F) above.

*Proof.* We just need to show the second equality; this is an application of Lemma 6.1 for  $c = r_0^\sharp(\mu_{01})$ ,  $d = \Lambda_{01}^\sharp$ ,  $b = \lambda_{12}^\sharp$ ,  $cd = \lambda_{01}^\sharp$ ,  $db = \Lambda_{01}^\sharp \lambda_{12}^\sharp$ ,  $cdb = \lambda_{02}^\sharp$  with their lifts chosen as in (A)-(F).  $\square$

We note that  $z(r_0^\sharp(\mu_{01}), \Lambda_{01}^\sharp) \in K_2(R_{01}[G])$ ; in fact, for almost all  $\eta_1$ , we have  $r_0^\sharp(\mu_{01}) \in \mathrm{GL}(R_1[G])$ ,  $\Lambda_{01}^\sharp \in \mathrm{GL}(R_1[G])$ , and so

$$(8.34) \quad \prod_{\eta_1} z(r_0^\sharp(\mu_{0\eta_1}), \Lambda_{0\eta_1}^\sharp) \in K'_2(\mathbb{A}_{\mathbb{P}^1, 01}[G])$$

which lies in the denominator of  $\mathrm{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$ .

**Lemma 8.8.** *We have an identity*

$$(8.35) \quad \begin{aligned} z(r_0^\sharp(\mu_{01}), r_0^\sharp(\mu_{12}) \Lambda_{02}^\sharp) z(r_0^\sharp(\mu_{12}), \Lambda_{02}^\sharp) &= z(r_0^\sharp(\mu_{01}) r_0^\sharp(\mu_{12}), \Lambda_{02}^\sharp) z(r_0^\sharp(\mu_{01}), r_0^\sharp(\mu_{12})) \\ &= z(r_0^\sharp(\mu_{02}), \Lambda_{02}^\sharp) z(r_0^\sharp(\mu_{01}), r_0^\sharp(\mu_{12})) \end{aligned}$$

where

$$\begin{aligned} z(r_0^\sharp(\mu_{01}), r_0^\sharp(\mu_{12}) \Lambda_{02}^\sharp) &= s_{02}(r_0^\sharp(\mu_{02}) \Lambda_{02}^\sharp) s_{012}(r_0^\sharp(\mu_{12}) \Lambda_{02}^\sharp)^{-1} s_{01}(r_0^\sharp(\mu_{01}))^{-1} \\ z(r_0^\sharp(\mu_{12}), \Lambda_{02}^\sharp) &= s_{012}(r_0^\sharp(\mu_{12}) \Lambda_{02}^\sharp) s_{02}(\Lambda_{02}^\sharp)^{-1} s_{012}(r_0^\sharp(\mu_{12}))^{-1} \\ z(r_0^\sharp(\mu_{01}) r_0^\sharp(\mu_{12}), \Lambda_{02}^\sharp) &= z(r_0^\sharp(\mu_{02}), \Lambda_{02}^\sharp) = s_{02}(r_0^\sharp(\mu_{02}) \Lambda_{02}^\sharp) s_{02}(\Lambda_{02}^\sharp)^{-1} s_{02}(r_0^\sharp(\mu_{02}))^{-1} \\ z(r_0^\sharp(\mu_{01}), r_0^\sharp(\mu_{12})) &= s_{02}(r_0^\sharp(\mu_{02})) s_{012}(r_0^\sharp(\mu_{12}))^{-1} s_{01}(r_0^\sharp(\mu_{01}))^{-1} \end{aligned}$$

with the lifts defined as in (A)-(F). Note here that  $r_0^\sharp(\mu_{02}) \Lambda_{02}^\sharp = \lambda_{02}^\sharp$  and therefore we take  $s_{02}(r_0^\sharp(\mu_{02}) \Lambda_{02}^\sharp) = s_{02}(\lambda_{02}^\sharp)$ . Recall that we also have  $r_0^\sharp(\mu_{01}) r_0^\sharp(\mu_{12}) = r_0^\sharp(\mu_{02})$ .

*Proof.* We just need to show the first identity. This follows from Lemma 6.1 applied to  $c = r_0^\sharp(\mu_{01})$ ,  $d = r_0^\sharp(\mu_{12})$ ,  $b = \Lambda_{02}^\sharp$ ,  $cd = r_0^\sharp(\mu_{01})r_0^\sharp(\mu_{12}) = r_0^\sharp(\mu_{02})$ ,  $db = r_0^\sharp(\mu_{12})\Lambda_{02}^\sharp$ ,  $cdb = r_0^\sharp(\mu_{01})r_0^\sharp(\mu_{12})\Lambda_{02}^\sharp = r_0^\sharp(\mu_{02})\Lambda_{02}^\sharp = \lambda_{02}^\sharp$  and their lifts as specified in (A)-(F).  $\square$

Note that  $z(r_0^\sharp(\mu_{02}), \Lambda_{02}^\sharp) \in K_2(R_{02}[G])$ . In fact, the following stronger statement is true. We chose lifts which are acceptable relative to some effective divisor  $\Delta$  on  $\mathbb{P}^1$  which contains all the vertical fibers over primes which divide the order of  $G$ . Therefore the lifts  $s_{02}(r_0^\sharp(\mu_{02}))$ ,  $s_{02}(\Lambda_{02}^\sharp)$  and  $s_{02}(r_0^\sharp(\mu_{02}) \cdot \Lambda_{02}^\sharp)$  lie in  $\text{St}(R_2[\Delta^{-1}][G])$ . This implies that  $z(r_0^\sharp(\mu_{02}), \Lambda_{02}^\sharp) \in K_2(R_2[\Delta^{-1}][G])$  for all choices of  $\eta_2$ . We conclude from this and Definition 2.2(b3) that

$$(8.36) \quad \prod_{\eta_2} z(r_0^\sharp(\mu_{0\eta_2}), \Lambda_{0\eta_2}^\sharp) \in K'_2(\mathbb{A}_{\mathbb{P}^1, 02}[G])$$

which lies in the denominator of  $\text{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$ .

**Corollary 8.9.** *There is a congruence*

$$(8.37) \quad z(\lambda^\sharp) \cdot z(r_0^\sharp(\mu))^{-1} \equiv \prod_{(\eta_1, \eta_2)} z(r_0^\sharp(\mu_{12}), \Lambda_{02}^\sharp)^{-1} \cdot z(\Lambda_{01}^\sharp, \lambda_{12}^\sharp)$$

in  $\text{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$ .

*Proof.* Because of (8.34), (8.33) gives the congruence

$$(8.38) \quad z(\lambda^\sharp) \equiv \prod_{(\eta_1, \eta_2)} z(r_0^\sharp(\mu_{01})\Lambda_{01}^\sharp, \lambda_{12}^\sharp) \equiv \prod_{(\eta_1, \eta_2)} z(r_0^\sharp(\mu_{01}), \Lambda_{01}^\sharp \lambda_{12}^\sharp) \cdot z(\Lambda_{01}^\sharp, \lambda_{12}^\sharp)$$

in  $\text{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$ . Because of (8.36), (8.35) gives the congruence

$$(8.39) \quad \begin{aligned} \prod_{(\eta_1, \eta_2)} z(r_0^\sharp(\mu_{01}), r_0^\sharp(\mu_{12})\Lambda_{02}^\sharp) \cdot z(r_0^\sharp(\mu_{12}), \Lambda_{02}^\sharp) &\equiv \prod_{(\eta_1, \eta_2)} z(r_0^\sharp(\mu_{01})r_0^\sharp(\mu_{12}), \Lambda_{02}^\sharp) \cdot z(r_0^\sharp(\mu_{01}), r_0^\sharp(\mu_{12})) \\ &\equiv z(r_0^\sharp(\mu)) \quad \text{in } \text{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G]). \end{aligned}$$

Multiply (8.38) by the inverse of (8.39). We conclude that  $z(\lambda^\sharp) \cdot z(r_0^\sharp(\mu))^{-1}$  is equal to

$$(8.40) \quad \prod_{(\eta_1, \eta_2)} z(r_0^\sharp(\mu_{01}), \Lambda_{01}^\sharp \lambda_{12}^\sharp) \cdot z(\Lambda_{01}^\sharp, \lambda_{12}^\sharp) \cdot z(r_0^\sharp(\mu_{01}), r_0^\sharp(\mu_{12})\Lambda_{02}^\sharp)^{-1} \cdot z(r_0^\sharp(\mu_{12}), \Lambda_{02}^\sharp)^{-1}.$$

in  $\text{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$ . We have  $\Lambda_{01}^\sharp \lambda_{12}^\sharp = r_0^\sharp(\mu_{12})\Lambda_{02}^\sharp$  by (8.18) and by our choices two terms on the right side of (8.40) cancel to give (8.37).  $\square$

We now expand the right hand side of (8.37). By definition we have

$$(8.41) \quad \begin{aligned} &z(r_0^\sharp(\mu_{12}), \Lambda_{02}^\sharp)^{-1} \cdot z(\Lambda_{01}^\sharp, \lambda_{12}^\sharp) = \\ &\left( s_{012}(r_0^\sharp(\mu_{12})) \cdot s_{02}(\Lambda_{02}^\sharp) \cdot s_{012}(r_0^\sharp(\mu_{12}) \cdot \Lambda_{02}^\sharp)^{-1} \right) \cdot (s_{012}(\Lambda_{01}^\sharp \lambda_{12}^\sharp) \cdot s_{12}(\lambda_{12}^\sharp)^{-1} \cdot s_{01}(\Lambda_{01}^\sharp)^{-1}). \end{aligned}$$

Notice that  $\Lambda_{01}^\sharp \lambda_{12}^\sharp = r_0^\sharp(\mu_{12})\Lambda_{02}^\sharp$  and  $s_{012}(r_0^\sharp(\mu_{12}) \cdot \Lambda_{02}^\sharp) = s_{012}(\Lambda_{01}^\sharp \lambda_{12}^\sharp)$  and so the middle terms on the right in (8.41) cancel. This and the expression for  $s_{012}(r_0^\sharp(\mu_{12}))$  in (8.27) show

$$(8.42) \quad \begin{aligned} &z(r_0^\sharp(\mu_{12}), \Lambda_{02}^\sharp)^{-1} \cdot z(\Lambda_{01}^\sharp, \lambda_{12}^\sharp) = \\ &s_{02}(\Lambda_{02}^\sharp) \cdot s_{12}(r_0^\sharp(\mu_{12})) \cdot s_{02}(\Lambda_{02}^\sharp)^{-1} \cdot s_{02}(\Lambda_{02}^\sharp) \cdot s_{12}(\lambda_{12}^\sharp)^{-1} \cdot s_{01}(\Lambda_{01}^\sharp)^{-1} = \\ &s_{02}(\Lambda_{02}^\sharp) \cdot s_{12}(r_0^\sharp(\mu_{12})) \cdot s_{12}(\lambda_{12}^\sharp)^{-1} \cdot s_{01}(\Lambda_{01}^\sharp)^{-1}. \end{aligned}$$

We now use the expression for  $s_{12}(\lambda_{12}^\#)$  in (8.26) to have

$$\begin{aligned}
 (8.43) \quad & z(r_0^\#(\mu_{12}), \Lambda_{02}^\#)^{-1} \cdot z(\Lambda_{01}^\#, \lambda_{12}^\#) = \\
 & s_{02}(\Lambda_{02}^\#) \cdot s_{12}(r_2^\#(\mu_{12})) \cdot \left( s_{12}(\Lambda_{12}^\#) \cdot s_{12}(r_2^\#(\mu_{12})) \right)^{-1} \cdot s_{01}(\Lambda_{01}^\#)^{-1} = \\
 & s_{02}(\Lambda_{02}^\#) \cdot s_{12}(\Lambda_{12}^\#)^{-1} \cdot s_{01}(\Lambda_{01}^\#)^{-1} = \\
 & z(\Lambda_{02}^\#, \Lambda_{01}^\#).
 \end{aligned}$$

Plugging this into the right hand side of (8.37) shows (8.32), and this completes the proof of Proposition 8.6.  $\square$

## 9. THE PROOF OF THE THEOREM; BUNDLES OVER $\mathbb{P}_{\mathbb{Z}}^1$

**9.a. Bundles over the projective line  $\mathbb{P}^1$ .** Let  $R$  be a commutative ring and let  $G$  be a finite group. Let  $\mathbb{P}^1 = \mathbb{P}_R^1$  be the projective line over  $\text{Spec}(R)$ . Thus  $\mathbb{P}^1$  is covered by two affine patches  $\mathbb{A}_0^1 = \text{Spec}(R[t])$  and  $\mathbb{A}_\infty^1 = \text{Spec}(R[t^{-1}])$  glued along  $\text{Spec}(R[t, t^{-1}])$ . If  $L$  is a module for  $R[G] \otimes_R R[t] = R[G][t]$  (resp.  $R[G] \otimes_R R[t^{-1}]$ ), let  $\tilde{L}$  be the corresponding sheaf of  $R[G]$ -modules on  $\mathbb{A}_0^1$  (resp.  $\mathbb{A}_\infty^1$ ). The following is a variation of a result of Horrocks.

**Theorem 9.1.** *Let  $R$  be a finite field or a Dedekind ring with finite residue fields. Let  $\mathcal{E}$  be an  $\mathcal{O}_{\mathbb{P}_R^1}[G]$ -bundle, i.e. a finitely generated locally free  $\mathcal{O}_{\mathbb{P}_R^1}[G]$ -module. Then there is a finitely generated locally free  $R[G]$ -module  $\mathcal{M}$  such that*

$$\mathcal{E}|_{\mathbb{A}_0^1} \simeq (\widetilde{\mathcal{M} \otimes_R R[t]}) \quad \text{and} \quad \mathcal{E}|_{\mathbb{A}_\infty^1} \simeq (\widetilde{\mathcal{M} \otimes_R R[t^{-1}]}).$$

*In particular, if  $R$  is a field or a local Dedekind ring then  $\mathcal{M}$  is a free  $R[G]$ -module.*

*Proof.* We first show that it will suffice to prove there is an isomorphism of  $R[G][t]$ -modules

$$(9.1) \quad \Gamma(\mathbb{A}_0^1, \mathcal{E}) \simeq \mathcal{M} \otimes_R R[t]$$

for some  $\mathcal{M}$  as in the Theorem. For then, on replacing  $\mathbb{A}_0^1$  by  $\mathbb{A}_\infty^1$ , we will have shown there is a finitely generated locally free free  $R[G]$ -module  $\mathcal{M}'$  such that  $\Gamma(\mathbb{A}_\infty^1, \mathcal{E}) \simeq \mathcal{M}' \otimes_R R[t]$ . This will imply there are  $R[G][t, t^{-1}]$ -module isomorphisms

$$\mathcal{M} \otimes_R R[t, t^{-1}] \simeq \Gamma(\mathbb{A}_0^1 \cap \mathbb{A}_\infty^1, \mathcal{E}) \simeq \mathcal{M}' \otimes_R R[t, t^{-1}].$$

By tensoring these isomorphisms with the  $R$ -algebra surjection  $R[t, t^{-1}] \rightarrow R$  which sends  $t$  to 1 we find that  $\mathcal{M} \simeq \mathcal{M}'$  as  $R[G]$ -modules, so Theorem 9.1 will follow.

Suppose now that  $R$  is a finite field. To prove (9.1) it will suffice to show that  $\Gamma(\mathbb{A}_0^1, \mathcal{E})$  is a free  $R[G][t]$ -module. Let  $r(R[G])$  be the quotient of  $R[G]$  by its maximal two-sided nilpotent ideal  $n(R[G])$ . Because  $\mathcal{E}$  is a locally free  $\mathcal{O}_{\mathbb{P}^1}[G]$ -module, we have an exact sequence of sheaves

$$(9.2) \quad 0 \rightarrow n(R[G]) \otimes_{R[G]} \mathcal{E} \rightarrow \mathcal{E} \rightarrow r(R[G]) \otimes_{R[G]} \mathcal{E} \rightarrow 0.$$

Since  $\mathbb{A}_0^1$  is affine, this gives a surjection

$$(9.3) \quad \Gamma(\mathbb{A}_0^1, \mathcal{E}) \rightarrow \Gamma(\mathbb{A}_0^1, r(R[G]) \otimes_{R[G]} \mathcal{E}).$$

The stalk  $\mathcal{E}_P$  of  $\mathcal{E}$  at each  $P \in \mathbb{A}_0^1$  is the localization  $\Gamma(\mathbb{A}_0^1, \mathcal{E})_P$  of  $\Gamma(\mathbb{A}_0^1, \mathcal{E})$  at  $P$  since  $\mathbb{A}_0^1$  is affine.

Let  $m$  be the rank of the locally free  $\mathcal{E}$ . Suppose we prove there is an isomorphism

$$(9.4) \quad (r(R[G])[t])^m \simeq \Gamma(\mathbb{A}_0^1, r(R[G]) \otimes_{R[G]} \mathcal{E})$$

of modules for  $r(R[G]) \otimes_R R[t] = r(R[G])[t]$ . Lift a set of  $m$  generators for the  $r(R[G])[t]$ -module  $\Gamma(\mathbb{A}_0^1, r(R[G]) \otimes_{R[G]} \mathcal{E})$  via the surjection (9.3). Because  $n(R[G])$  is nilpotent in (9.2), this produces  $m$  elements of  $\Gamma(\mathbb{A}_0^1, \mathcal{E})$  which generate the stalk  $\mathcal{E}_P = \Gamma(\mathbb{A}_0^1, \mathcal{E})_P$  at each point  $P \in \mathbb{A}_0^1$ . This gives a homomorphism  $\psi : (R[G][t])^m \rightarrow \mathcal{E}$  which localizes at each  $P$  to an isomorphism of locally free  $\mathcal{O}_{\mathbb{P}^1, P}[G]$ -modules. Thus  $\psi$  is an isomorphism, so when  $R$  is finite we are reduced to showing (9.4).

The ring  $r(R[G])$  is semi-simple and is thus isomorphic to a finite direct sum  $\oplus_i R_i$  of simple  $R$ -algebras  $R_i$ . Since  $R$  is finite,  $R_i$  is isomorphic to a matrix algebra  $\text{Mat}_{n_i}(k_i)$  for some finite extension  $k_i$  of  $R$  and some integer  $n_i \geq 1$ . Thus  $r(R[G]) \otimes_{R[G]} \mathcal{E}$  is isomorphic to  $\oplus_i E_i$  where  $E_i$  is a rank  $m$  locally free  $R_i$ -module on  $\mathbb{P}^1 = \mathbb{P}_R^1$ . Therefore to show (9.4), it will suffice to show that  $\Gamma(\mathbb{A}_0^1, E_i)$  is a free rank  $m$  module for  $R_i \otimes_R R[t] = R_i[t]$ . There is a Morita equivalence between the category of modules for  $R_i = \text{Mat}_{n_i}(k_i)$  and the category of vector spaces over  $k_i$ . This implies that it will suffice to show that a locally free rank  $m$  sheaf  $T_i$  of  $k_i$ -modules on  $\mathbb{P}_R^1$  has the property that

$$(9.5) \quad \Gamma(\mathbb{A}_0^1, T_i) \simeq (k_i \otimes_R R[t])^m = (k_i[t])^m$$

as  $k_i[t]$ -modules. Here  $T_i$  corresponds to a rank  $m$  vector bundle on  $k_i \otimes_R \mathbb{P}_R^1 = \mathbb{P}_{k_i}^1$ , so the isomorphism (9.5) follows from [26, Theorem 1]. This completes the proof when  $R$  is a finite field.

Suppose now that  $R$  is a discrete valuation ring with finite residue field  $k$  and uniformizer  $\pi$ . Since  $\mathbb{A}_0^1$  is affine and  $\mathcal{E}$  is a locally free  $\mathcal{O}_{\mathbb{P}_R^1}[G]$ -module, we have an exact sequence

$$0 \rightarrow \pi \cdot \Gamma(\mathbb{A}_0^1, \mathcal{E}) \rightarrow \Gamma(\mathbb{A}_0^1, \mathcal{E}) \rightarrow \Gamma(\mathbb{A}_0^1, k \otimes_R \mathcal{E}) \rightarrow 0$$

where  $\Gamma(\mathbb{A}_0^1, k \otimes_R \mathcal{E}) \simeq \Gamma(k \otimes \mathbb{A}_0^1, k \otimes_R \mathcal{E})$  is a free  $k[G][t]$ -module by what has already been shown for finite fields. On lifting generators and using Nakayama's Lemma we see that  $\Gamma(\mathbb{A}_0^1, \mathcal{E})$  is a free  $R[G][t]$ -module.

Finally, suppose  $R$  is a Dedekind ring with finite residue fields. Following Quillen [44] we will call an  $R[G][t]$ -module  $M$  *extended* if it is isomorphic to  $N \otimes_R R[t]$  for some locally free  $R[G]$ -module  $N$ . This implies  $N$  is isomorphic to  $M/tM$ . By what has already been shown for discrete valuation rings, for each maximal ideal  $\mathfrak{m}$  of  $R$ , the localization

$$\Gamma(\mathbb{A}_0^1, \mathcal{E})_{\mathfrak{m}} = \Gamma(R_{\mathfrak{m}} \otimes_R \mathbb{A}_0^1, R_{\mathfrak{m}} \otimes_R \mathcal{E})$$

is an extended  $R_{\mathfrak{m}}[G][t]$ -module. To complete the proof it will suffice to show that  $M = \Gamma(\mathbb{A}_0^1, \mathcal{E})$  is an extended  $R[G][t]$ -module. We briefly sketch how this follows from Quillen's patching Lemma ([44, Theorem 1]).

First observe that since we do know that  $M_{\mathfrak{m}}$  is extended, when we let  $N = M/tM$ , the localization  $N_{\mathfrak{m}}$  is a locally free  $R_{\mathfrak{m}}[G]$ -module of rank equal to the locally free rank  $m$  of

$\mathcal{E}$ . Since  $R$  is a Dedekind ring and  $\mathfrak{m}$  ranges over all maximal ideals of  $R$ , this implies  $N$  is a locally free  $R[G]$ -module of rank  $m$ .

As in [44], let  $S$  be the set of  $f \in R$  such that  $M_f$  is extended as a module for  $R_f[G][t]$ . It will suffice to show that  $1 \in S$ . The argument in the first part of the proof of Theorem 1 of [44] shows that it will suffice to show that if  $f_0, f_1 \in S$  and  $Rf_0 + Rf_1 = R$  then  $1 \in S$ . Suppose  $f \in S$ . We have

$$(9.6) \quad \begin{aligned} \mathrm{Hom}_{R_f[G][t]}(N \otimes_R R_f[t], N \otimes_R R_f[t]) &= \mathrm{Hom}_{R[G]}(N, N \otimes_R R_f[t]) \\ &= R_f[t] \otimes_R \mathcal{A} = \mathcal{A}_f[t] \end{aligned}$$

when  $\mathcal{A} = \mathrm{End}_{R[G]}(N)$ . The remainder of the proof of Theorem 1 in [44] now applies because  $\mathcal{A}$  is allowed to be non-commutative in [44, Lemma 1].  $\square$

9.a.1. Suppose that  $M_0$  is a finitely generated locally free  $R[G]$ -module and  $\gamma$  an element of the group  $\mathrm{Aut}(M_0 \otimes_R R[t, t^{-1}])$  of the  $R[G][t, t^{-1}]$ -linear automorphisms of  $M_0 \otimes_R R[t, t^{-1}]$ . Then, glueing  $M_0 \otimes_R R[t]$  and  $M_0 \otimes_R R[t^{-1}]$  by using  $\gamma$  provides a finitely generated locally free  $\mathcal{O}_{\mathbb{P}_R^1}[G]$ -module  $\mathcal{E} = \mathcal{E}(M_0, \gamma)$ . By Theorem 9.1, if  $R$  is a finite field or a Dedekind ring, every finitely generated locally free  $\mathcal{O}_{\mathbb{P}_R^1}[G]$ -module  $\mathcal{E}$  is of this form. We then call  $(M_0, \gamma)$  “Horrocks data” associated to  $\mathcal{E}$ . When  $M_0 \simeq R[G]^n$  is free, we can identify  $\mathrm{GL}_n(R[G][t, t^{-1}])$  with the group  $\mathrm{Aut}(M_0 \otimes_R R[t, t^{-1}])$  by sending  $g$  to  $\gamma_g$  given by  $m \mapsto m \cdot g^{-1}$ . Then we write the Horrocks data  $(M_0, \gamma_g)$  simply as  $(M_0, g)$ .

#### 9.b. The adelic Riemann-Roch theorem over $\mathbb{P}_{\mathbb{Z}}^1$ .

9.b.1. We first show a special case of our main result over  $\mathbb{P}^1 = \mathbb{P}_{\mathbb{Z}}^1$ . Write  $f : \mathbb{P}_{\mathbb{Z}}^1 \rightarrow S = \mathrm{Spec}(\mathbb{Z})$  for the structure morphism. We continue to assume that the group algebra  $\mathbb{Q}[G]$  splits in the sense of Definition 2.9.

**Theorem 9.2.** *Suppose  $\mathcal{F}$  is an  $\mathcal{O}_{\mathbb{P}^1}[G]$ -bundle of rank  $m$  on  $\mathbb{P}^1 = \mathbb{P}_{\mathbb{Z}}^1$  which satisfies:*

- (a) *The reduced Euler characteristics  $\bar{\chi}(\mathbb{P}^1, \mathcal{F})_{\mathbb{Q}} \in K_0(\mathbb{Q}[G])$ , and  $\bar{\chi}(\mathbb{P}^1, \mathcal{F})_{\mathbb{Z}_p} \in K_0(\mathbb{Z}_p[G])$ , for all primes  $p$ , are trivial;*
- (b) *The  $\mathbb{Z}[G]$ -module obtained by pulling back  $\mathcal{F}$  along  $\mathrm{Spec}(\mathbb{Z}) \rightarrow \mathbb{P}^1$  given by  $t = 1$  is stably free.*

*Then*

- i) *The sheaf  $\mathcal{F}$  has an (adelic) elementary structure. Therefore, the first Chern class  $c_1(\mathcal{F})$  is trivial in  $\mathrm{CH}_{\mathbb{A}}^1(\mathbb{P}^1[G])$  and the second Chern class  $c_2(\mathcal{F})$  is defined in  $\mathrm{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$ .*
- ii) *We have the Riemann-Roch identity*

$$(9.7) \quad \bar{\chi}^P(\mathbb{P}^1, \mathcal{F}) = -f_*(c_2(\mathcal{F}))$$

$$\text{in } \mathrm{Cl}(\mathbb{Z}[G]) = K_0^{\mathrm{red}}(\mathbb{Z}[G]) = \mathrm{CH}_{\mathbb{A}}^1(\mathrm{Spec}(\mathbb{Z})[G]).$$

*Proof.* Recall that, by definition  $\bar{\chi}(\mathbb{P}^1, \mathcal{F})_{\mathbb{Q}} = \chi(\mathbb{P}^1, \mathcal{F})_{\mathbb{Q}} - \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}[G]^m)_{\mathbb{Q}}$  and similarly for  $\bar{\chi}(\mathbb{P}^1, \mathcal{F})_{\mathbb{Z}_p}$ . Notice that by our constructions, the statements (i) and (ii) are true for  $\mathcal{F}$  if and only if they are true for the bundle  $\mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^1}[G]^n$  for some  $n \geq 0$ . Also  $\bar{\chi}(\mathbb{P}^1, \mathcal{F})_{\mathbb{Q}} = \bar{\chi}(\mathbb{P}^1, \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^1}[G]^n)_{\mathbb{Q}}$ ,  $\bar{\chi}(\mathbb{P}^1, \mathcal{F})_{\mathbb{Z}_p} = \bar{\chi}(\mathbb{P}^1, \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^1}[G]^n)_{\mathbb{Z}_p}$ . By Theorem 9.1, assumption (b) and these observations, we may assume that the sheaf  $\mathcal{F}$  is given by Horrocks data



$(\mathbb{Z}[G]^m, \gamma_g)$  where  $g \in \mathrm{GL}_m(\mathbb{Z}[G][t, t^{-1}])$ . Hence, we can write  $\mathcal{F} = \mathcal{E}(L_0 \cdot g^{-1})$  where  $L_0 = \mathbb{Z}[G][t]^m$  and in our notation (see §3.c.2, §3.d),

$$\mathcal{V}_g = \delta(L_0 \cdot g^{-1}) - \delta(L_0) = \det(R\Gamma(\mathbb{P}^1, \mathcal{F})) - \det(R\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}[G]^m)).$$

This shows that assumption (a) implies that the matrix  $g \in \mathrm{GL}_m(\mathbb{Z}[G][t, t^{-1}])$  actually belongs to  $\mathrm{GL}'_m(\mathbb{Q}[G][t, t^{-1}])$  and to  $\mathrm{GL}'_m(\mathbb{Z}_p[G][t, t^{-1}])$  for all primes  $p$ . Denote by  $[g]$  the class of  $g$  in  $K_1(\mathbb{Z}[G][t, t^{-1}])$ . In the next paragraph, we will denote  $g$  by  $g_{\mathbb{Q}}$  when we consider it as an element of  $\mathrm{GL}'_m(\mathbb{Q}[G][t, t^{-1}])$  and by  $g_p$  when we consider it as an element of  $\mathrm{GL}'_m(\mathbb{Z}_p[G][t, t^{-1}])$ . Recall now that

$$\mathrm{Cl}(\mathbb{Z}[G]) = K_0^{\mathrm{red}}(\mathbb{Z}[G]) = \mathrm{CH}_{\mathbb{A}}^1(\mathrm{Spec}(\mathbb{Z})[G]) = \frac{\prod_p' K_1(\mathbb{Q}_p[G])}{K_1(\mathbb{Q}[G]) \cdot \prod_p K_1(\mathbb{Z}_p[G])^b}.$$

To calculate a  $K_1$ -idele in  $\prod_p' K_1(\mathbb{Q}_p[G])$  which maps to  $\bar{\chi}^P(\mathbb{P}^1, \mathcal{F})$  under this map we argue as follows. Choose trivializations

$$\alpha_{\mathbb{Q}} : [0] \xrightarrow{\sim} (\mathcal{V}_g)_{\mathbb{Q}} = (\det(R\Gamma(\mathbb{P}^1, \mathcal{F})) - \det(R\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}[G]^m)))_{\mathbb{Q}}$$

$$\alpha_{\mathbb{Z}_p} : [0] \xrightarrow{\sim} (\mathcal{V}_g)_{\mathbb{Z}_p} = (\det(R\Gamma(\mathbb{P}^1, \mathcal{F})) - \det(R\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}[G]^m)))_{\mathbb{Z}_p}$$

and consider, for each prime  $p$ , the element  $\alpha_p^{-1} \cdot \alpha_{\mathbb{Q}}$  in the group  $\mathrm{Aut}_{V(\mathbb{Q}_p[G])}([0]) = K_1(\mathbb{Q}_p[G])$ . The  $K_1$ -idele  $(\alpha_p^{-1} \cdot \alpha_{\mathbb{Q}})_p$  represents  $\bar{\chi}^P(\mathbb{P}^1, \mathcal{F})$ . By the definition of the central extensions, the elements  $\alpha_{\mathbb{Q}}, \alpha_p$  correspond to lifts  $\tilde{g}_{\mathbb{Q}} = w_{\mathbb{Q}}(g_{\mathbb{Q}})$ ,  $\tilde{g}_p = w_p(g_p)$  of  $g_{\mathbb{Q}}, g_p$  in

$$1 \rightarrow K_1(\mathbb{Q}[G]) \rightarrow \mathcal{H}(\mathbb{Q}[G][t, t^{-1}]^m) \rightarrow \mathrm{GL}'_m(\mathbb{Q}[G][t, t^{-1}]) \rightarrow 1,$$

$$1 \rightarrow K_1(\mathbb{Z}_p[G]) \rightarrow \mathcal{H}(\mathbb{Z}_p[G][t, t^{-1}]^m) \rightarrow \mathrm{GL}'_m(\mathbb{Z}_p[G][t, t^{-1}]) \rightarrow 1.$$

We can now write (recall  $g_p = g_{\mathbb{Q}} = g$  in  $\mathrm{GL}'_m(\mathbb{Q}_p[G][t, t^{-1}])$ )

$$\begin{aligned} (9.8) \quad \tilde{g}_{\mathbb{Q}} \cdot \tilde{g}_p^{-1} &= (g_{\mathbb{Q}}, \alpha_{\mathbb{Q}})(g_p, \alpha_p)^{-1} = (g_{\mathbb{Q}}, \alpha_{\mathbb{Q}})(g_p^{-1}, \alpha_p^{-g_p^{-1}}) \\ &= (1, (\alpha_p^{-g_p^{-1}})^{g_{\mathbb{Q}}} \cdot \alpha_{\mathbb{Q}}) = (1, \alpha_p^{-1} \cdot \alpha_{\mathbb{Q}}) = \alpha_p^{-1} \cdot \alpha_{\mathbb{Q}}. \end{aligned}$$

Hence,  $\alpha_p^{-1} \cdot \alpha_{\mathbb{Q}} = \tilde{g}_{\mathbb{Q}} \cdot \tilde{g}_p^{-1}$  with the product calculated in  $\mathcal{H}(\mathbb{Q}_p[G][t, t^{-1}])$  and we conclude that

$$(9.9) \quad \bar{\chi}^P(\mathbb{P}^1, \mathcal{F}) = \prod_p (\tilde{g}_{\mathbb{Q}} \cdot \tilde{g}_p^{-1})$$

in  $K_0^{\mathrm{red}}(\mathbb{Z}[G]) = (\prod_p' K_1(\mathbb{Q}_p[G]))/K_1(\mathbb{Q}[G]) \cdot \prod_p K_1(\mathbb{Z}_p[G])^b$ .

To show the Riemann-Roch identity, we will express the element in the right hand side of (9.9) as the negative of the push-down of the second Chern class of  $\mathcal{F}$ . We continue by giving first some preliminaries.

9.b.2. Recall the set-up and definitions of §2.d. Let  $R$  be an integral domain with fraction field  $N$  of characteristic 0. Let  $N^c$  be an algebraic closure of  $N$ . Consider the base change

$$K_1(R[t, t^{-1}][G]) \rightarrow K_1(N^c[t, t^{-1}][G]).$$

Using Lemma 2.6 we can see that the kernel of this is equal to  $SK_1(R[t, t^{-1}][G])$ . Recall that the Bass-Heller-Swan theorem gives a homomorphism

$$b_R : K_1(R[t, t^{-1}][G]) \rightarrow K_0(R[G]) \times K_1(R[G]).$$

The base change  $K_1(R[G]) \rightarrow K_1(R[t, t^{-1}][G])$  splits the second projection. The map  $b$  is an isomorphism when  $R = N$  is a field of characteristic zero. An explicit description of  $b$  for the algebraically closed  $N^c$  is as follows: Using Morita equivalence and by taking determinants we obtain an isomorphism

$$(9.10) \quad K_1(N^c[G][t, t^{-1}]) \xrightarrow{\sim} \text{Hom}(R_G, N^c[t, t^{-1}]^\times)$$

where  $R_G$  is the group of  $N^c$ -valued characters of  $G$ . Since  $(N^c[t, t^{-1}])^\times = t^\mathbb{Z} \cdot (N^c)^\times$  the target can be written

$$\text{Hom}(R_G, t^\mathbb{Z}) \times \text{Hom}(R_G, (N^c)^\times) \xrightarrow{\sim} K_0(N^c[G]) \times K_1(N^c[G])$$

and  $b_{N^c}$  is the resulting composition.

**Lemma 9.3.** *Suppose  $(N[G]^m, g)$  are Horrocks data for an  $\mathcal{O}_{\mathbb{P}_N^1}[G]$ -bundle  $\mathcal{E}$  on  $\mathbb{P}_N^1$ . Denote by  $[g]$  the class of  $g$  in  $K_1(N[t, t^{-1}][G])$ . Then the component of  $b_{N^c}([g])$  in  $\text{Hom}(R_G, t^\mathbb{Z}) = K_0(N^c[G])$  is given by the character function*

$$\chi \mapsto t^{\deg((\mathcal{E} \otimes_{N^c} V_{\bar{\chi}})^G)},$$

where  $V_\psi$  is a  $N^c[G]$ -module with character  $\psi$ . As a result, the component of  $b_N([g])$  in  $K_0(N[G])$  is equal to the reduced Euler characteristic  $\bar{\chi}(\mathbb{P}_N^1, \mathcal{E}) = \chi(\mathbb{P}_N^1, \mathcal{E}) - \chi(\mathbb{P}_N^1, \mathcal{O}_{\mathbb{P}_N^1}[G]^m)$  in  $K_0(N[G])$ .

*Proof.* The second part of the statement follows from the first part and the (usual) Riemann-Roch theorem on the curve  $\mathbb{P}_{N^c}^1$ . To show the first part is enough to observe that the degree of a vector bundle obtained by gluing as above is given by the valuation of the determinant of the transition (gluing) matrix at  $t = 0$ .  $\square$

9.b.3. We now continue with the proof of Theorem 9.2. Since  $g$  is in  $\text{GL}'_m(\mathbb{Q}[G][t, t^{-1}])$  Lemma 9.3 and the above discussion implies that there is  $\kappa_{\mathbb{Q}} \in K_1(\mathbb{Q}[G])$  with  $\text{Det}([g])^{-1} = \text{Det}(\kappa_{\mathbb{Q}})$ . Similarly, since  $g$  is in  $\text{GL}'_m(\mathbb{Z}_p[G][t, t^{-1}])$  and  $K_0(\mathbb{Z}_p[G]) \subset K_0(\mathbb{Q}_p[G])$  we obtain that there is  $\kappa_p \in K_1(\mathbb{Z}_p[G])$  such that  $\text{Det}([g])^{-1} = \text{Det}(\kappa_p)$ . Lift  $\kappa_{\mathbb{Q}}, \kappa_p$  to  $z_{\mathbb{Q}} \in \mathbb{Q}[G]^\times$ ,  $z_p \in \mathbb{Z}_p[G]^\times$ , and consider the elements  $g'_{\mathbb{Q}} = z_{\mathbb{Q}} \cdot g \in \text{GL}(\mathbb{Q}[G][t, t^{-1}])$ ,  $g'_p = z_p \cdot g \in \text{GL}(\mathbb{Z}_p[G][t, t^{-1}])$ . For these elements we have

$$[g'_{\mathbb{Q}}] \in SK_1(\mathbb{Q}[G][t, t^{-1}]), \quad [g'_p] \in SK_1(\mathbb{Z}_p[G][t, t^{-1}]).$$

Since by Morita equivalence and Lemma 2.6,  $\mathrm{SK}_1(\mathbb{Q}[G][t, t^{-1}]) = (0)$ , this shows that  $g'_\mathbb{Q}$  is in  $E(\mathbb{Q}[G][t, t^{-1}])$ . Consider the image of  $[g'_p]$  in  $\mathrm{SK}_1(\mathbb{Z}_p[G]\{\{t\}\})$ . By Corollary 2.13, the natural homomorphism

$$\mathrm{SK}_1(\mathbb{Z}_p[G]\langle\langle t^{-1} \rangle\rangle) \rightarrow \mathrm{SK}_1(\mathbb{Z}_p[G]\{\{t\}\}),$$

where  $\mathbb{Z}_p\langle\langle t^{-1} \rangle\rangle$  is the  $p$ -adic completion of  $\mathbb{Z}_p[t^{-1}]$ , is surjective. Therefore, for each  $p$ , we can find an element  $h_p \in \mathrm{GL}'(\mathbb{Z}_p[G]\langle\langle t^{-1} \rangle\rangle)$  with  $[h_p] = [g'_p]^{-1}$  in  $\mathrm{SK}_1(\mathbb{Z}_p[G]\{\{t\}\}) \hookrightarrow \mathrm{K}_1(\mathbb{Z}_p[G]\{\{t\}\})$ . Notice that for those  $p$  which do not divide the order of  $G$  and where  $z_\mathbb{Q}$  is a unit, we can take  $z_p = z_\mathbb{Q}$  and  $h_p = 1$ .

We will now show how to choose (stably) Parshin bases  $f_{\eta_i}$  for  $\mathcal{F}$  which provide us with an (adelic) elementary structure. For simplicity, we will write  $\hat{\mathcal{O}}_{\eta_i}$ ,  $\hat{\mathcal{O}}_{\eta_i\eta_j}$ , instead of  $\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_i}$ ,  $\hat{\mathcal{O}}_{\mathbb{P}^1, \eta_i\eta_j}$ , etc. The Horrocks description for  $\mathcal{F}$  provide us bases  $e_{\eta_i}$  for  $\hat{\mathcal{F}}_{\eta_i}$  which are determined by fixing a basis of  $M_0 \simeq \mathbb{Z}[G]^m$ . Denote by 0 the (unique) generic point of  $\mathbb{P}^1$ . Denote by  $1_H$  the generic point of the divisor  $H = \{t = 0\}$ , and for each prime  $p$ , denote by  $1_p$  the generic point of the fiber of  $p$ . We also denote by  $2_p$  the unique closed point  $t = 0$  in characteristic  $p$  which is the intersection of  $1_H$  and  $1_p$ . The Horrocks gluing description implies that there is a basis  $e_0 = \{e_0^h\}_{h=1}^m$  over the generic point such that that  $e_\eta = g^{-1}e_0$  if  $\eta$  is on  $1_H$  and  $e_\eta = e_0$  otherwise. This implies the following values for the transition matrices  $\lambda_{\eta_i\eta_j}$  (recall  $e_{\eta_i} = \lambda_{\eta_i\eta_j}e_{\eta_j}$ ):

$$(9.11) \quad \lambda_{0\eta_1} = \begin{cases} g, & \text{if } \eta_1 = 1_H, \\ 1, & \text{if } \eta_1 \neq 1_H. \end{cases}$$

$$(9.12) \quad \lambda_{\eta_1\eta_2} = \begin{cases} 1, & \text{if } \eta_2 \neq 2_p, \\ 1, & \text{if } \eta_2 = 2_p, \eta_1 = 1_H, \\ g, & \text{if } \eta_2 = 2_p, \eta_1 \neq 1_H, \end{cases}$$

All the other values are determined from these and the cocycle condition. We now give different bases  $f_{\eta_i}$  by  $f_0 = z_\mathbb{Q} \cdot e_0$  and

$$f_{\eta_1} = \begin{cases} e_{1_H}, & \text{if } \eta_1 = 1_H \\ z_p h_p \cdot e_{1_p}, & \text{if } \eta_1 = 1_p \\ z_\mathbb{Q} \cdot e_{\eta_1}, & \text{if } \eta_1 \notin \{H, 1_p \text{ for all } p\} \end{cases}, \quad f_{\eta_2} = \begin{cases} e_{2_p}, & \text{if } \eta_2 = 2_p \\ z_p h_p \cdot e_{\eta_2}, & \text{if } \eta_2 \neq 2_p \text{ in char. } p. \end{cases}$$

These give the following values for the transition matrices  $\theta_{\eta_i\eta_j}$  with respect to  $f_{\eta_i}$ :

$$(9.13) \quad \theta_{0\eta_1} = \begin{cases} 1, & \text{if } \eta_1 \notin \{1_H, 1_p \text{ for all } p\} \\ z_\mathbb{Q} \cdot g, & \text{if } \eta_1 = 1_H \\ z_\mathbb{Q} h_p^{-1} z_p^{-1}, & \text{if } \eta_1 = 1_p \end{cases}$$

$$(9.14) \quad \theta_{\eta_1 \eta_2} = \begin{cases} 1, & \text{if } \eta_1 = 1_H, \\ 1, & \text{if } \eta_1 = 1_p, \eta_2 \neq 2_p \\ z_{\mathbb{Q}} \cdot g, & \text{if } \eta_2 = 2_p, \eta_1 \neq 1_H, 1_p \text{ for all } p \\ z_p h_p \cdot g, & \text{if } \eta_1 = 1_p, \eta_2 = 2_p \\ z_{\mathbb{Q}} h_p^{-1} z_p^{-1}, & \text{if } \eta_1 \text{ is horizontal, } \eta_1 \neq 1_H, \eta_2 \neq 2_p \text{ in characteristic } p. \end{cases}$$

We now verify that the matrices  $\theta_{\eta_i \eta_j}$  are elementary for all pairs  $\eta_i, \eta_j$ . By our construction,  $\text{Det}(\theta_{\eta_i \eta_j}) = 1$ . (Notice for example that  $\text{Det}(z_{\mathbb{Q}}) = \text{Det}(z_p) = \text{Det}(g)^{-1}$ .) Observe that: by Morita equivalence and the fact that  $\text{SK}_1$  is trivial for commutative local rings, we know that  $\text{SK}_1(\hat{\mathcal{O}}_{0\eta_1}[G]) = (0)$ , and that  $\text{SK}_1(\hat{\mathcal{O}}_{\eta_1 \eta_2}[G]) = (0)$  if  $\eta_1$  is horizontal; and by Proposition 2.7 and Morita equivalence we know that  $\text{SK}_1(\hat{\mathcal{O}}_{0\eta_2}[G]) = (0)$ . The only thing left to check is that  $\theta_{1_p 2_p} = z_p h_p g$  has trivial image in  $\text{SK}_1(\hat{\mathcal{O}}_{1_p 2_p}[G]) = \text{SK}_1(\mathbb{Z}_p[G] \{\{t\}\})$ . This follows from our choice of  $z_p, h_p$  above.

Hence, the above completes the proof of part (i) of the statement of Theorem 9.2.

It now remains to show part (ii) which is the Riemann-Roch identity.

Notice that the elements  $\theta_{\eta_i \eta_j}$  satisfy the conclusion of part (a) of Proposition 6.4 for the divisor  $D$  which is the union of  $1_H$  with the fibers  $1_p$  over the finite list  $T$  of primes  $p$  which either divide the order of  $G$  or are such that  $z_{\mathbb{Q}} \in \mathbb{Q}[G]^{\times}$  does not belong to  $\mathbb{Z}_p[G]^{\times}$ . Set  $Q = \prod_{p \in T} p$ . Enlarge  $T$  and the corresponding divisor  $D$  to ensure that the group  $\text{SK}_1(\mathbb{Z}[Q^{-1}][t, t^{-1}][G])$  is trivial. (We can do this since, by our assumption, for sufficiently large  $Q$ ,  $\mathbb{Z}[Q^{-1}][G]$  is a product of matrix rings with entries in principal ideal domains; we can then follow the same arguments as in §9.b.2.)

We will now show how to choose lifts  $\tilde{\theta}_{\eta_i \eta_j}$  as in Proposition 6.4 that can be used to calculate the adelic second Chern class according to the recipe in Definition 6.5.

**Proposition 9.4.** *There are choices of lifts  $\tilde{\theta}_{\eta_i \eta_j}$  of  $\theta_{\eta_i \eta_j}$  so that  $\tilde{\theta}_{\eta_0 \eta_1} \in \text{St}(\hat{\mathcal{O}}_{\eta_1}[D^{-1}][G])$ ,  $\tilde{\theta}_{\eta_1 \eta_2} \in \text{St}(\hat{\mathcal{O}}_{\eta_1 \eta_2}[G])$ , and  $\tilde{\theta}_{\eta_0 \eta_2} \in \text{St}(\hat{\mathcal{O}}_{\eta_2}[D^{-1}][G])$ , so that for the element  $z(\tilde{\theta})$  of Proposition 6.4 (b) we have:*

- (i)  $z(\tilde{\theta})_{(\eta_0, \eta_1, \eta_2)} = 1$ , unless  $\eta_1 = 1_p$  and  $\eta_2 = 2_p$ ,
- (ii)  $z(\tilde{\theta})_{(0, 1_p, 2_p)} = 1$ , if  $p$  does not divide the order of the group  $G$  and is such that  $z_{\mathbb{Q}} \in \mathbb{Q}[G]^{\times}$  belongs to  $\mathbb{Z}_p[G]^{\times}$ .

*Proof.* Let us prove (i) first. We will consider  $z(\tilde{\theta})_{(\eta_0, \eta_1, \eta_2)}$  for  $\eta_2$  in characteristic  $p$ . We suppose we do not have  $\eta_1 = 1_p, \eta_2 = 2_p$ . Then all cases of such triples  $(\eta_0, \eta_1, \eta_2)$  have similar structure: namely, one of the  $\theta_{\eta_i \eta_j} = 1$  and so the remaining two transition maps  $\theta_{\eta_a \eta_b}$  are equal (up to inversion) to a value that we denote  $\theta$ ; there are three relevant rings  $\hat{\mathcal{O}}_{\eta_1}[D^{-1}]$ ,  $\hat{\mathcal{O}}_{\eta_1 \eta_2}$ , and  $\hat{\mathcal{O}}_{\eta_2}[D^{-1}]$  and the common value  $\theta$  belongs to the intersection of two of them. In all cases we shall identify a subring  $R$  of this intersection with the property that  $\theta \in E(R[G])$ ; we can then use a lift  $\tilde{\theta} \in \text{St}(R[G])$  twice in computing  $z(\tilde{\theta})_{(\eta_0, \eta_1, \eta_2)}$  to get the value 1.

*Case 1: Horizontal case  $\eta_1 = 1_H$ .*

Here there is only one situation to consider; namely,  $\eta_2 = 2_p$ ; then  $\theta_{1_H \eta_2} = 1$ ,  $\theta_{0 \eta_2} = z_{\mathbb{Q}} g_{\mathbb{Q}} = \theta_{0 1_H}$ . We know that  $z_{\mathbb{Q}} g_{\mathbb{Q}} \in \text{SL}(\mathbb{Z}[Q^{-1}][t, t^{-1}][G])$ . Since  $\text{SK}_1(\mathbb{Z}[Q^{-1}][t, t^{-1}][G]) =$

$\{1\}$ , we have  $z_{\mathbb{Q}}g_{\mathbb{Q}} \in E(\mathbb{Z}[Q^{-1}][t, t^{-1}][G])$ . Note that

$$(9.15) \quad \mathbb{Z}[Q^{-1}][t, t^{-1}] \rightarrow \hat{\mathcal{O}}_{1_H}[D^{-1}], \quad \mathbb{Z}[Q^{-1}][t, t^{-1}] \rightarrow \hat{\mathcal{O}}_{\eta_2}[D^{-1}].$$

We can therefore use the surjection  $\text{St}(\mathbb{Z}[Q^{-1}][t, t^{-1}][G]) \rightarrow E(\mathbb{Z}[Q^{-1}][t, t^{-1}][G])$  to find a common lift of both  $\theta_{1_H\eta_2}$  and  $\theta_{0\eta_2}$ ; the corresponding  $z(\tilde{\theta})_{(0,1_H,\eta_2)}$  is then trivial. In other words, here we take in the above sketch  $R = \mathbb{Z}[Q^{-1}][t, t^{-1}]$ .

*Case 2: Horizontal case  $\eta_1 \neq 1_H$ .*

Here  $\theta_{\eta_0\eta_1} = 1$  and there are two subcases to consider.

Subcase (a):  $\eta_2 \neq 2_p$ . Then  $\theta_{\eta_1\eta_2} = z_{\mathbb{Q}}h_p^{-1}z_p^{-1} = \theta_{\eta_0\eta_2}$ . We have  $z_{\mathbb{Q}}h_p^{-1}z_p^{-1} = 1$ , if  $p \notin T$ , and  $z_{\mathbb{Q}}h_p^{-1}z_p^{-1} \in \text{SL}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p \langle \langle t^{-1} \rangle \rangle [G]) = E(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p \langle \langle t^{-1} \rangle \rangle [G])$  if  $p \in T$  (cf. Lemma 2.14). The situation is trivial if  $p \notin T$ . If  $p \in T$ ,  $\hat{\mathcal{O}}_{\eta_2}[D^{-1}]$  and  $\hat{\mathcal{O}}_{\eta_1\eta_2}$  are the two relevant rings; we can then take  $R$  to be  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p \langle \langle t^{-1} \rangle \rangle$  and proceed as before to get  $z(\tilde{\theta})_{(\eta_0,\eta_1,\eta_2)} = 1$ .

Subcase (b):  $\eta_2 = 2_p$ . Then  $\theta_{\eta_1\eta_2} = z_{\mathbb{Q}}g_{\mathbb{Q}} = \theta_{\eta_0\eta_2}$ ,  $\hat{\mathcal{O}}_{\eta_2}[D^{-1}] = \mathbb{Z}_p((t))[Q^{-1}]$  and  $\hat{\mathcal{O}}_{\eta_1\eta_2}$  are the two relevant rings; here we can work with  $R = \mathbb{Z}_p[Q^{-1}][t, t^{-1}]$ . Notice here that since  $\eta_1 \neq 1_H$ ,  $t$  is invertible in  $\hat{\mathcal{O}}_{\eta_1\eta_2}$ .

*Case 3: Vertical case when  $\eta_1 = 1_p$ .*

Here there is only the case  $\eta_2 \neq 2_p$  (since the case  $\eta_2 = 2_p$  is excluded). Then  $\theta_{1_p\eta_2} = 1$  and  $\theta_{\eta_01_p} = z_{\mathbb{Q}}h_p^{-1}z_p^{-1} = \theta_{\eta_0\eta_2}$ . The two relevant rings are  $\hat{\mathcal{O}}_{\eta_1}[D^{-1}] = \hat{\mathcal{O}}_{\eta_1}[t^{-1}, Q^{-1}]$ ,  $\hat{\mathcal{O}}_{\eta_2}[D^{-1}] = \mathbb{Z}_p[[t - \eta_2]][t^{-1}, Q^{-1}]$ ; Since  $t$  is a unit in  $\mathbb{Z}_p[[t - \eta_2]]$ , here we can work with  $R = \mathbb{Z}_p \langle \langle t^{-1} \rangle \rangle [Q^{-1}]$ .

To prove (ii) we observe that for a prime  $p$  that does not divide the order of the group and with  $z_{\mathbb{Q}}$  a unit at  $p$ , we have taken  $z_p = z_{\mathbb{Q}}$ ,  $h_p = 1$ . Then  $\theta_{01_p} = 1$ ,  $\theta_{1_p2_p} = \theta_{02_p} = z_{\mathbb{Q}} \cdot g = z_p \cdot g$ . Part (ii) then follows by a similar argument as above by using the ring  $R = \mathbb{Z}_p[Q^{-1}][t, t^{-1}]$ .

This completes the proof of Proposition 9.4.  $\square$

By Proposition 9.4 a representative for the pushdown  $f_*(c_2(\mathcal{F}))$  of the second adelic Chern class  $c_2(\mathcal{F})$  of  $\mathcal{F}$  is given by the idele in  $\prod'_p K_1(\mathbb{Q}_p[G])$  whose component at  $p$  is given by the push-down  $f_*$  of

$$(9.16) \quad z(\tilde{\theta})_{(0,1_p,2_p)} = s_{02}(z_{\mathbb{Q}}g_{\mathbb{Q}})s_{12}(z_ph_pg)^{-1}s_{01}(z_{\mathbb{Q}}h_p^{-1}z_p^{-1})^{-1}.$$

Here, for clarity, in the right hand side, we use the symbol  $s_{12}$  to denote the lift of an element in the Steinberg group  $\text{St}(\hat{\mathcal{O}}_{\eta_1\eta_2}[G]) = \text{St}(\mathbb{Z}_p\{\{t\}\}[G])$ , the symbol  $s_{02}$  to denote a lift in the Steinberg group  $\text{St}(\hat{\mathcal{O}}_{\eta_2}[D^{-1}][G]) = \text{St}(\mathbb{Z}_p((t))[Q^{-1}][G])$ , and the symbol  $s_{01}$  to denote a lift in the Steinberg group  $\text{St}(\hat{\mathcal{O}}_{\eta_1}[D^{-1}][G])$ . The product is taken in  $\text{St}(\hat{\mathcal{O}}_{\eta_0\eta_1\eta_2}[G]) = \text{St}(\mathbb{Q}_p\{\{t\}\}[G])$ .

Recall that with the above notation we write  $w_{\mathbb{Q}}(g_{\mathbb{Q}}) = (g_{\mathbb{Q}}, \alpha_{\mathbb{Q}})$ ,  $w_p(g_p) = (g_p, \alpha_p)$ . The desired result will follow if we show

$$f_*(z(\tilde{\theta})_{(0,1_p,2_p)}) = \alpha_{\mathbb{Q}}^{-1} \alpha_p \kappa_{\mathbb{Q}}^{-1} \kappa_p$$

with  $\kappa_{\mathbb{Q}} \in K_1(\mathbb{Q}[G])^b$ ,  $\kappa_p \in K_1(\mathbb{Z}_p[G])^b$ . We now evaluate the pushdown  $f_*(z(\tilde{\theta})_{(0,1_p,2_p)})$  by working with each of the three right hand terms in equation (9.16). Recall that the push-down is defined via the inverse of the homomorphisms  $\partial$ ,  $\hat{\partial}$  and in particular,  $f_*(z(\tilde{\theta})_{(0,1_p,2_p)}) = \hat{\partial}(z(\tilde{\theta})_{(0,1_p,2_p)})^{-1}$ .

1)  $s_{02}(z_{\mathbb{Q}}g_{\mathbb{Q}})$ : By the above we know that

$$z_{\mathbb{Q}}g_{\mathbb{Q}} \in \mathrm{SL}(\mathbb{Z}[Q^{-1}][t, t^{-1}][G]) \subset \mathrm{E}(\mathbb{Z}[Q^{-1}] \otimes \mathbb{Z}((t))[G]) \subset \mathrm{E}(\hat{\mathcal{O}}_{02_p}[G]).$$

We let  $\mathcal{H}(\mathbb{Z}[Q^{-1}] \otimes \mathbb{Z}((t))[G])$  denote the pullback of  $\mathcal{H}(\mathbb{Q}((t))[G])$  along

$$\mathrm{GL}'(\mathbb{Z}[Q^{-1}] \otimes \mathbb{Z}((t))[G]) \subset \mathrm{GL}'(\mathbb{Q}((t))[G]).$$

We may now compute using the following diagram

$$(9.17) \quad \begin{array}{ccc} \mathrm{E}(\mathbb{Z}[Q^{-1}] \otimes \mathbb{Z}((t))[G]) & \xrightarrow{s_Q} & \mathrm{St}(\mathbb{Z}[Q^{-1}] \otimes \mathbb{Z}((t))[G]) \\ \downarrow \text{inclusion} & & \downarrow \partial \\ \mathrm{GL}'(\mathbb{Z}[Q^{-1}] \otimes \mathbb{Z}((t))[G]) & \xrightarrow{w_Q} & \mathcal{H}(\mathbb{Z}[Q^{-1}] \otimes \mathbb{Z}((t))[G]) \end{array}$$

which commutes up to an element of  $K_1(\mathbb{Q}[G])$  (see (3.12)). Here  $w_Q$  is a set-theoretic section of the  $\mathcal{H}$ -sequence that is compatible with the natural splitting of  $\mathcal{H}(\mathbb{Q}((t))[G])$  over  $\mathrm{GL}(\mathbb{Q}[G])$ . We then get the equality

$$(9.18) \quad \hat{\partial}(s_{02}(z_{\mathbb{Q}}g_{\mathbb{Q}})) = \partial(s_Q(z_{\mathbb{Q}}g_{\mathbb{Q}})) = \kappa_{\mathbb{Q}}w_Q(z_{\mathbb{Q}}g_{\mathbb{Q}}) = \kappa_{\mathbb{Q}}z_{\mathbb{Q}}w_Q(g_{\mathbb{Q}})$$

for some  $\kappa_{\mathbb{Q}} \in K_1(\mathbb{Q}[G])$ .

2)  $s_{12}(z_ph_pg_p)$ : From the above we know that

$$z_p \in \mathbb{Z}_p[G]^\times, \quad h_p \in \mathrm{SL}(\mathbb{Z}_p\langle\langle t^{-1} \rangle\rangle[G]), \quad g_p \in \mathrm{GL}'(\mathbb{Z}_p[t, t^{-1}][G]).$$

Thus all terms lie in  $\mathrm{GL}'(\mathbb{Z}_p\{\{t\}\}[G]) = \mathrm{GL}'(\hat{\mathcal{O}}_{1_p2_p}[G])$  and by construction the product of these three elements lies in  $\mathrm{E}(\mathbb{Z}_p\{\{t\}\}[G])$ . We now compute  $\hat{\partial}(s_{12}(z_ph_pg_p))$  using the analogous diagram to (9.17) above for the ring  $\mathbb{Z}_p\{\{t\}\}[G]$ , which commutes up to an element of  $K_1(\mathbb{Z}_p[G])$ . Using the fact that  $\mathcal{H}(\mathbb{Z}_p\{\{t\}\}[G])$  splits naturally over  $\mathrm{GL}'(\mathbb{Z}_p\langle\langle t^{-1} \rangle\rangle[G])$ , we obtain the equality

$$(9.19) \quad \hat{\partial}(s_{12}(z_ph_pg_p)) = \kappa_pw_p(z_ph_pg_p) = \kappa_pz_ph_pw_p(g_p)$$

for some  $\kappa_p \in K_1(\mathbb{Z}_p[G])$ .

3)  $s_{01}(z_{\mathbb{Q}}h_p^{-1}z_p^{-1})$ : If  $p \notin T$  then this term is trivial. We assume that  $p$  is in  $T$  so that  $\mathbb{Z}_p[Q^{-1}] = \mathbb{Q}_p$ . From the above work we know that

$$z_{\mathbb{Q}} \in \mathbb{Z}[Q^{-1}][G]^\times, \quad h_p \in \mathrm{SL}(\mathbb{Z}_p\langle\langle t^{-1} \rangle\rangle[G]), \quad z_p \in \mathbb{Z}_p[G]^\times.$$

Thus all terms lie in  $\mathrm{GL}(\mathbb{Q}_p\{t^{-1}\}[G])$ . By construction their product lies in  $\mathrm{SL}(\mathbb{Q}_p\{t^{-1}\}[G])$ . (Recall,  $\mathbb{Q}_p\{t^{-1}\} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p\langle\langle t^{-1} \rangle\rangle$  is the free Tate algebra.)

Using Lemma 2.14 we deduce that  $z_{\mathbb{Q}}h_p^{-1}z_p^{-1} \in \mathrm{E}(\mathbb{Q}_p\{t^{-1}\}[G])$ . Define  $\hat{\mathcal{H}}(\mathbb{Q}_p\{t^{-1}\}[G])$  to be the pullback of  $\hat{\mathcal{H}}(\mathbb{Q}_p\{\{t\}\}[G])$  along the inclusion

$$\mathrm{GL}(\mathbb{Q}_p\{t^{-1}\}[G]) \hookrightarrow \mathrm{GL}^*(\mathbb{Q}_p\{\{t\}\}[G]).$$

By §3.e.4 we have a natural splitting  $w$  of the resulting central extension of  $\mathrm{GL}(\mathbb{Q}_p\{t^{-1}\}[G])$  which agrees with the natural splittings over  $\mathbb{Q}[G]^\times$ ,  $\mathbb{Z}_p[G]^\times$ ,  $\mathrm{SL}(\mathbb{Z}_p\langle\langle t^{-1}\rangle\rangle[G])$  that we have used in cases (1) and (2) above. We can also restrict this extension along  $\mathrm{E}(\mathbb{Q}_p\{t^{-1}\}[G])$ . The fact that this last extension of  $\mathrm{E}(\mathbb{Q}_p\{t^{-1}\}[G])$  splits also follows from Corollary 4.5, since the Steinberg sequence for  $\mathbb{Q}_p\{t^{-1}\}[G]$  is the universal central extension of the perfect group  $\mathrm{E}(\mathbb{Q}_p\{t^{-1}\}[G])$ . The universality gives

$$\hat{\partial} : \mathrm{St}(\mathbb{Q}_p\{t^{-1}\}[G]) \rightarrow \hat{\mathcal{H}}(\mathbb{Q}_p\{\{t\}\}[G])$$

which then has to be equal to the composition  $w \circ \pi$  where  $\pi$  is the natural homomorphism from the Steinberg group to the elementary group of  $\mathbb{Q}_p\{t^{-1}\}[G]$ . Thus we get

$$(9.20) \quad \hat{\partial}(s_{01}(z_{\mathbb{Q}}h_p^{-1}z_p^{-1})) = w(z_{\mathbb{Q}}h_p^{-1}z_p^{-1}) = z_{\mathbb{Q}}h_p^{-1}z_p^{-1}.$$

In summary from (9.18), (9.19) and (9.20), we see that the expression

$$\hat{\partial}(z(\tilde{\theta})_{(0,1_p,2_p)}) = \hat{\partial}(s_{02}(z_{\mathbb{Q}}g_{\mathbb{Q}}))\hat{\partial}(s_{12}(z_ph_pg_p))^{-1}\hat{\partial}(s_{01}(z_{\mathbb{Q}}h_p^{-1}z_p^{-1}))^{-1}$$

is equal to

$$\begin{aligned} & z_{\mathbb{Q}}w_{\mathbb{Q}}(g_{\mathbb{Q}})(z_ph_pw_p(g_p))^{-1}(z_{\mathbb{Q}}h_p^{-1}z_p^{-1})^{-1}\kappa_{\mathbb{Q}}\kappa_p^{-1} \\ &= z_{\mathbb{Q}}w_{\mathbb{Q}}(g_{\mathbb{Q}})w_p(g_p)^{-1}h_p^{-1}z_p^{-1}z_ph_pz_{\mathbb{Q}}^{-1}\kappa_{\mathbb{Q}}\kappa_p^{-1} \\ &= w_{\mathbb{Q}}(g_{\mathbb{Q}})w_p(g_p)^{-1}\kappa_{\mathbb{Q}}\kappa_p^{-1}. \end{aligned}$$

By (9.8) above this is equal to  $\alpha_p^{-1}\alpha_{\mathbb{Q}}\kappa_{\mathbb{Q}}\kappa_p^{-1}$ . Since  $f_*(z(\tilde{\theta})_{(0,1_p,2_p)}) = \hat{\partial}(z(\tilde{\theta})_{(0,1_p,2_p)})^{-1}$  we obtain  $f_*(z(\tilde{\theta})_{(0,1_p,2_p)}) = \kappa_p\kappa_{\mathbb{Q}}^{-1}\alpha_{\mathbb{Q}}^{-1}\alpha_p$  which completes our proof.  $\square$

9.b.4. In this last paragraph we show an adelic Riemann-Roch theorem for general bundles on  $\mathbb{P}_{\mathbb{Z}}^1$ . Suppose  $\mathcal{E}$  is a  $\mathcal{O}_{\mathbb{P}^1}[G]$ -bundle of rank  $n$  on  $\mathbb{P}_{\mathbb{Z}}^1$  with “Horrocks data”  $(M_0, g)$ . We first give (somewhat ad-hoc) definitions of  $ch_1(\mathcal{E}) \cap H$  and  $ch_2(\mathcal{E})$  in  $\mathrm{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$ .

Suppose that  $(a_p)_p \in \prod'_p \mathrm{K}_1(\mathbb{Q}_p[G])$  is a  $\mathrm{K}_1$ -idele representing the class of the locally free module  $M_0$  in  $\mathrm{K}_0^{\mathrm{red}}(\mathbb{Z}[G]) = \mathrm{CH}_{\mathbb{A}}^1(S[G])$ . By Remark 5.3, this class coincides with the first Chern class  $c_1(\tilde{M}_0)$  of the corresponding  $\mathcal{O}_S[G]$ -sheaf over  $S = \mathrm{Spec}(\mathbb{Z})$  as defined in §5.a. If  $D = m \cdot H$  is a divisor on  $\mathbb{P}_{\mathbb{Z}}^1$  which is a multiple of the hyperplane section  $H = \{t = 0\}$ , we denote by  $ch_1(\mathcal{E}) \cap (m \cdot H) = c_1(\mathcal{E}) \cap (m \cdot H)$  the element in  $\mathrm{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$  represented by the  $\mathrm{K}_2$ -idele which is equal to  $\{a_p, t^m\}$  at the triples  $(0, 1_p, 2_p)$  for all  $p$ , and 1 everywhere else. (Here, since  $t$  is central, we can make sense of the symbol  $\{a_p, t^m\}$  in  $\mathrm{K}_2(\mathbb{Q}_p\{\{t\}\}[G])$ .)

We can find a locally free ideal  $I \subset \mathbb{Z}[G]$  with  $M_0 \simeq I \oplus \mathbb{Z}[G]^{n-1}$  (see for example [48]). Find also an ideal  $J \subset \mathbb{Z}[G]$  such that  $I \oplus J \simeq \mathbb{Z}[G]^2$ . Let  $d = \deg(\det(\mathcal{E}))/\#G$ . We let

$$(9.21) \quad \mathcal{F} = \mathcal{E} \oplus f^*\tilde{J} \oplus \mathcal{O}_{\mathbb{P}^1}(-d)[G].$$

We can see that the sheaf  $\mathcal{F}$  satisfies the assumptions (a), (b) of Theorem 9.2. To define  $ch_2(\mathcal{E}) \in \mathrm{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G])$ , we now set

$$(9.22) \quad ch_2(\mathcal{E}) := ch_2(\mathcal{F}) = -2c_2(\mathcal{F}) \quad \text{in } \mathrm{CH}_{\mathbb{A}}^2(\mathbb{P}^1[G]).$$



**Remark 9.5.** Notice here that we are lacking a direct and general definition of  $ch_2(\mathcal{E})$  that applies to all locally free  $\mathcal{O}_Y[G]$ -sheaves on a arbitrary arithmetic surface  $Y$ . However, over  $\mathbb{P}^1$ , the above definition makes sense. Indeed, it is a reasonable expectation that the two summands  $f^*\tilde{J}$ ,  $\mathcal{O}_{\mathbb{P}^1}(-d)[G]$ , should have trivial  $ch_2$ 's. Indeed,  $f^*\tilde{J}$  is pulled-back from the 1-dimensional  $S = \text{Spec}(\mathbb{Z})$ , while  $ch_2(\mathcal{O}_{\mathbb{P}^1}(-d)) = 0$  in the non-equivariant setting. Similarly, one can ask if it is possible to define an intersection pairing

$$CH_{\mathbb{A}}^1(Y[G]) \times CH^1(Y) \rightarrow CH_{\mathbb{A}}^2(Y[G]) ; \quad (c, D) \mapsto c \cap D$$

by capping with a local generator of the divisor  $D$  when  $Y$  is a regular arithmetic surface. Such a construction would give a more natural interpretation of  $ch_1(\mathcal{E}) \cap H = c_1(\mathcal{E}) \cap H$  given above.

**Theorem 9.6.** (*Adelic Riemann-Roch theorem for  $\mathbb{P}^1$* ) Let  $\mathcal{E}$  be a locally free coherent  $\mathcal{O}_{\mathbb{P}^1}[G]$ -module  $\mathcal{E}$  of rank  $n$  on  $\mathbb{P}_{\mathbb{Z}}^1$ . Assume  $\mathbb{Q}[G]$  splits as in Definition 2.9. Then

$$(9.23) \quad \chi^P(\mathbb{P}^1, \mathcal{E}) - \chi^P(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}[G]^n) = f_*((ch_1(\mathcal{E}) + \frac{ch_2(\mathcal{E})}{2}) \cap (1 + \frac{c_1(T_{\mathbb{P}^1/S})}{2}))$$

in  $Cl(\mathbb{Z}[G]) = K_0^{\text{red}}(\mathbb{Z}[G]) = CH_{\mathbb{A}}^1(S[G])$ .

In this expression, the numerators of both fractions are canonically divisible by 2. (By (9.22) and  $T_{\mathbb{P}^1/S} \simeq \mathcal{O}_{\mathbb{P}^1}(2)$  so that  $c_1(T_{\mathbb{P}^1/S})/2 = H$ .) The precise meaning of the right hand side is  $f_*(ch_1(\mathcal{E}) \cap H) - f_*(c_2(\mathcal{F}))$ . We choose to write the result in this (ambiguous) way to resemble the expression in the classical Grothendieck-Riemann-Roch theorem.

*Proof.* Recall that  $\mathcal{F} = \mathcal{E} \oplus f^*\tilde{J} \oplus \mathcal{O}_{\mathbb{P}^1}(-d)[G]$  with  $[J] + [M_0] = 0$  in  $K_0^{\text{red}}(\mathbb{Z}[G])$ . Hence, we have  $\bar{\chi}^P(\mathbb{P}^1, \mathcal{F}) = \bar{\chi}^P(\mathbb{P}^1, \mathcal{E}) - [M_0]$ . The right hand side of (9.23) is interpreted as

$$f_*(-c_2(\mathcal{F}) + c_1(\mathcal{E}) \cap H).$$

Since  $f_*(\{a_p, t\}) = a_p$ , this is equal to  $-f_*(c_2(\mathcal{F})) + [M_0]$ . Therefore, (9.23) reduces to

$$(9.24) \quad \bar{\chi}^P(\mathbb{P}^1, \mathcal{F}) = -f_*(c_2(\mathcal{F}))$$

in  $K_0^{\text{red}}(\mathbb{Z}[G]) = CH_{\mathbb{A}}^1(S[G])$ . Here  $\mathcal{F}$  satisfies conditions (a), (b) of Theorem 9.2, and the result follows from loc. cit.  $\square$

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